Winter Term, 2016

Name: SOLUTIONS

WILFRID LAURIER UNIVERSITY

Waterloo, Ontario

Mathematics 350A – Real Analysis

Midterm Test 1 – February 12, 2016

Instructor:

Dr. Yuming Chen

Time Allowed: *50 minutes*

Total Value: *45 marks*

Number of Pages: *5 plus cover page*

Instructions:

Non-programmable, non-graphing calculators are permitted. No other aids are allowed.

Check that your test paper has no missing, blank, or illegible pages.

Answer in the spaces provided. ***Please note that questions are printed on both sides of the test pages.***

Show all your work. Insufficient justification will result in a loss of marks.

Write your student number in the space provided on the next page.

⇝ **Student Number:** ______________

1. Let X and Y be nonempty sets.

[3 marks] (a) State the definition that X **is equivalent to** Y.

Solution. We say that X is equivalent to Y and write $X \approx Y$ if there exists a one-to-one function f from X onto Y.

[5 marks] (b) If X is equivalent to Y, show that $\mathscr{P}(X)$ is also equivalent to $\mathscr{P}(Y)$, where $\mathscr{P}(A)$ is the collection of all subsets of A for a set A.

Proof. Since X is equivalent to Y, there exists a one-to-one function f from X onto Y. Define $g : \mathscr{P}(X) \to \mathscr{P}(Y)$ by

$$g(A) = \{f(a) | a \in A\} \qquad \text{for } A \in \mathscr{P}(A).$$ 2

We claim that g is a one-to-one and onto function and hence $\mathscr{P}(X)$ is equivalent to $\mathscr{P}(Y)$. First, we show that g is one-to-one. Suppose $g(A_1) = g(A_2)$. If $A_1 \neq A_2$, then there exists $a_1 \in A_1$ but $a_1 \notin A_2$ or there exists $a_2 \in A_2$ but $a_2 \notin A_1$. Consider the first case as the second can be dealt with similarly. Then $f(a_1) \in g(A_1) = g(A_2)$ and there exists $a_2 \in A_2$ such that $f(a_1) = f(a_2)$. It follows that $a_1 = a_2$ since f is one-to-one, a contradiction. Therefore, $A_1 = A_2$, which implies that g is one-to-one. 2 Next, we show g is onto. Let $B \in \mathscr{P}(Y)$. Since f is onto, we have $g(f^{-1}(B)) = B$. This completes the proof.

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[3 marks] 2. (a) State the definition of a Cauchy sequence of real numbers.

Solution. Let $\{a_n\}$ be a sequence of real numbers. We say that $\{a_n\}$ is Cauchy if for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|a_n - a_m| < \varepsilon \qquad \text{for } n,\, m \geq N.$$

[4 marks] (b) According to the definition in part (a), prove that the sequence $\{1 + \frac{1}{\sqrt{n}}\}$ is a Cauchy sequence.

Proof. For any $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that $N > \frac{4}{\varepsilon^2}$. (2) Then for n, $m \geq N$, we have

$$\begin{aligned}
\left|\left(1+\frac{1}{\sqrt{n}}\right) - \left(1+\frac{1}{\sqrt{m}}\right)\right| &= \left|\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{m}}\right| \\
&\leq \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{m}} \\
&\leq \frac{2}{\sqrt{N}} && (1) \\
&< \varepsilon, && (1)
\end{aligned}$$

that is, $\{1 + \frac{1}{\sqrt{n}}\}$ is Cauchy.

[3 marks] 3. (a) State the Bolzano-Weierstrass theorem.

Solution. The Bolzano-Weierstrass theorem says that every bounded real sequence has a convergent subsequence.

[4 marks] (b) Let $\{a_n\}$ be a bounded sequence of real numbers. Use the Bolzano-Weierstrass theorem to show that there is a subsequence $\{a_{n_k}\}$ ($k \in \mathbb{N}$) which is convergent and such that for every k, n_k is a perfect square.

Proof. First note that the subsequence $\{a_{n^2}\}$ is a bounded real sequence since $\{a_n\}$ is. Then by the Bolzano-Weierstrass theorem, there exists a convergent subsequence $\{a_{n_k}\}$ of $\{a_{n^2}\}$, which is also a subsequence of $\{a_n\}$. Obviously, $\{a_{n_k}\}$ satisfies the requirement.

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[*5 marks*] 4. Show that the sequence $\{(1+\frac{1}{n})^{n+1}\}$ is a decreasing sequence.

Proof. Let $a_n = (1+\frac{1}{n})^{n+1}$. Then for $n \in \mathbb{N}$,

$$\begin{aligned}
\frac{a_{n+1}}{a_n} &= \frac{(1+\frac{1}{n+1})^{n+2}}{(1+\frac{1}{n})^{n+1}} \\
&= \left(1+\frac{1}{n+1}\right)\left(\frac{1+\frac{1}{n+1}}{1+\frac{1}{n}}\right)^{n+1} \\
&= \left(1+\frac{1}{n+1}\right)\left(1-\frac{1}{(n+1)^2}\right)^{n+1} && 2 \\
&\le \left(1+\frac{1}{(n+1)^2}\right)^{n+1}\left(1-\frac{1}{(n+1)^2}\right)^{n+1} && \text{(here we used the binomial formula)} \quad 2 \\
&= \left(1-\frac{1}{(n+1)^4}\right)^{n+1} \\
&< 1, && 1
\end{aligned}$$

or $a_{n+1} < a_n$. Therefore, $\{a_n\}$ is a decreasing sequence.

[*6 marks*] 5. Let $\{a_n\}$ be a sequence of non-negative real numbers and suppose that $\sum_{n=1}^{\infty} a_n$ diverges. Prove that $\sum_{n=1}^{\infty} \sqrt{a_n}$ also diverges.
[**Hint:** Use by way of contradiction.]

Proof. By way of contradiction, suppose that $\sum_{n=1}^{\infty} a_n$ converges. Then $\lim_{n\to\infty} \sqrt{a_n} = 0$. It follows that there exists $N \in \mathbb{N}$ such that

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$$\sqrt{a_n} < 1 \qquad \text{when } n \ge N.$$

Thus

$$a_n \le \sqrt{a_n} \qquad \text{for } n \ge N. \quad 2$$

By the Comparison Test, $\sum_{n=1}^{\infty} a_n$ converges, a contradiction. 2

[5 marks] 6. Give an example of two infinite series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ that satisfy all of the following three properties. Justify your answer.

(i) Both series diverge.

(ii) $a_n \neq b_n$ for all $n \in \mathbb{N}$.

(iii) $\sum_{n=1}^{\infty}(a_n - b_n)$ converges.

Solution. Let $a_n = \frac{1}{n} + \frac{1}{n^2}$ and $b_n = \frac{1}{n}$ for $n \in \mathbb{N}$. Then $a_n \neq b_n$ for all $n \in \mathbb{N}$. Since

$$a_n, b_n \geq \frac{1}{n} \qquad \text{for all } n \in \mathbb{N}$$

and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ diverge by the Comparison Test. Note that $\sum_{n=1}^{\infty}(a_n - b_n) = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent. This shows that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ satisfy the three properties.

[7 marks] 7. Suppose that $m \leq a_n \leq M$ for all $n \in \mathbb{N}$, where m and M are positive real numbers. Use the Root Test to find the radius of convergence of the power series $\sum_{n=1}^{\infty} n a_n x^n$. Justify your answer.

Solution. Note that for $n \in \mathbb{N}$, we have

$$\sqrt[n]{nm} \leq \sqrt[n]{na_n} \leq \sqrt[n]{nM}. \quad 1$$

Since $\lim_{n\to\infty} \sqrt[n]{n} = \lim_{n\to\infty} \sqrt[n]{m} = \lim_{n\to\infty} \sqrt[n]{M} = 1$, we have $\lim_{n\to\infty} \sqrt[n]{nm} = \lim_{n\to\infty} \sqrt[n]{nM} = 1$. By the Squeeze Theorem, $\lim_{n\to\infty} \sqrt[n]{na_n} = 1$. Then the radius of convergence of the power series $\sum_{n=1}^{\infty} n a_n x^n$ is $R = \frac{1}{\lim_{n\to\infty} \sqrt[n]{na_n}} = 1$. 2

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