COMP. 233.

ASSIGN. 2.

1. Three balls are randomly selected without replacement from an urn containing 20 balls numbered 1 through 20. If we bet that at least one of the drawn balls has a number greater than or equal to 17, what is the probability that we win the bet?

2. A complex Engineering System is composed of five subsystems. Each of these subsystems is independent of the other four. The lifetime, in hours, of each of these subsystems is given by:

$$f(x) = \begin{cases} 0 & \text{if } x \leq 100 \\ 100/x^2 & \text{if } x > 100 \end{cases}$$

   What is the probability that exactly 2 out of the 5 subsystems in the Complex System, will have to be replaced within the first 150 hrs. of Operation?

3. Compute the Cumulative Distribution Function for the function f(x) in problem 2.

   A. $P(X \leq 100)$?
   B. $P(X \leq 130)$?
   C. $P(X > 140)$?
   D. $P(110 \leq X \leq 120)$?

4. Buses arrive at a specified stop at 15-minute intervals starting at 7:00 A.M. That is, they arrive at 7:00, 7:15, 7:30, 7:45,.....
   A passenger arrives at the stop with probability f(x):

$$f(x) = \begin{cases} 1/30 & \text{if } 0 \leq x \leq 30 \\ 0 & \text{otherwise} \end{cases}$$

where X is the R.V. denoting the number of minutes passed 7:00 that the passenger arrives at the stop.

As is evident, the passenger only arrives at the stop between 7:00 – 7:30 A.M.
What is the probability that the passenger:

A. Waits less than 5 mins. for a bus?

B. Waits more than 10 mins. for a bus?

5. A fair coin is flipped three times.
   Let the R.V. X: denote the number of heads obtained in the first two flips.
   Y: denote the number of heads obtained in the first three flips.

   Compute:
   A. $p(x, y)$?

   B. $P(Y = 1)$?

   C. $P(X = 0)$?

6. Let X denote the R.V. that equals the number of tails minus the number of heads when N fair coins are flipped.

   A. What is the expected value of X?

   B. Prove that the variance of X equals N.

7. M balls are to be distributed randomly into N bins. What is the expected number of balls which will fall into the first bin?

8. Divide, at random, a horizontal line segment of length five.

   Let X: length of the left-hand part.

Given: Cumulative Distribution Function of X.

$$F(x) = \begin{cases} 0 & \text{if } x \le 0 \\ x/5 & \text{if } 0 < x \le 5 \\ 1 & \text{if } x > 5 \end{cases}$$

Compute:

A. P.D.F. $f(x)$?

B. $E[X]$?

C. $E[5 - X]$?

D. Does $E[X(5 - X)] = E[X]\,E[5 - X]$?

9. Let X_1, X_2 be two independent R.V.'s, each having P.M.F.:

$$p(x) = \begin{cases} x/6 & \text{if } x = 1, 2, 3 \\ 0 & \text{elsewhere} \end{cases}$$

Let $X = X_1 + X_2$.

A. Compute the Moment Generating Function of X.

B. From part A, compute $E[X]$.

C. From part A, compute $\mathrm{Var}(X)$.

10. Let X be a R.V. taking on only positive values. The mean of X is 0, and the variance of X is 4.

Compute $P(X \ge 3)$ as best you could.
Explain your reasoning.

11. Discrete random variables X and Y , whose values are positive integers, have the joint probability mass function:

$$p(x,y) = 2^{-x-y} .$$

A. Determine the marginal probability mass functions p(x) and p(y).
B. Are X and Y independent?
C. Determine E[X], E[Y], and E[XY].

12. The distribution function of a real random variable X is given by:

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ x/2 & \text{if } 0 \leq x < 1 \\ 2/3 & \text{if } 1 \leq x < 2 \\ 11/12 & \text{if } 2 \leq x < 3 \\ 1 & \text{if } x \geq 3 \end{cases}$$

(a) Plot this distribution function.
(b) What is $P(X > ½)$?
(c) What is $P(2 < X \leq 4)$?
(d) What is $P(X < 3)$?
(e) What is $P(X = 1)$?

13. Suppose that random variables X and Y have a joint density function given by:

$$f(x,y) = \begin{cases} x + y & if\ 0 \leq x \leq 1,\ 0 \leq y \leq 1 \\ 0 & otherwise \end{cases} .$$

(a) Find the density functions of X and Y , f(x) and f(y).
(b) Find E[X] and Var(Y).

14. A machine makes machine parts that are screened (inspected 100 percent) before being shipped. The instrument measuring the diameter of a part is such that it is difficult to read between 1 and 4/3 . After the screening process takes place, the measured diameter has density:

$$f(x) = \begin{cases} kx^2 & if\ 0 \le x \le 1 \\ 1 & if\ 1 < x \le 4/3 \\ 0 & otherwise \end{cases} .$$

(a) Find the value of k.
(b) What fraction of the parts will fall outside the twilight zone (range between 0 and 1)?
(c) Find the mean and variance of this random variable.

15. From past experience, a professor knows that the test score of students taking a final examination is a random variable with mean 65.

(a) Give an upper bound on the probability that a student's test score will exceed 75.
(b) Suppose in addition the professor knows that the variance of a student's test score is equal to 30. What can be said about the probability that a student will score between 55 and 75?
(c) How many students would have to take the examination so as to ensure, with probability at least 0.8, that the class average would be within 5 of 65?

16. A communications channel transmits the digits 0 and 1. However, due to static, the digit transmitted is incorrectly received with probability 0.2. Suppose that we want to transmit an important message consisting of one binary digit. To reduce the chance of error, we transmit 00000 instead of 0 and 11111 instead of 1.

    A. If the receiver of the message uses "majority" decoding, what is the probability that the message will be incorrectly decoded?

    B. What independence assumptions are you making? (By majority decoding we mean that the message is decoded as "0" if there are at least three zeros in the message received and as "1" otherwise.)

17. Approximately 20,000 marriages took place in Quebec last year. Estimate the probability that for at least one of these couples:

    (a) both partners were born on April 1.
    (b) both partners celebrated their birthday on the same day of the year.

    Assume that each person's birthday is equally likely to be any of the 365 days of the year.

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COMP. 233.

Assign. 2 Solutions.

1. Let the R.V. X denote the largest number selected from the three drawn balls. X is then a discrete R.V. taking on values 3, 4, 5, ..., 20.

$$p(X=i) = \frac{\binom{i-1}{2}}{\binom{20}{3}} = \frac{{}_{(i-1)}C_2}{{}_{20}C_3} \qquad i = 3, 4, 5, \ldots, 20$$

$$\therefore P(X \geq 17) = P\big((X=17) \cup (X=18) \cup (X=19) \cup (X=20)\big)$$

$$= P(X=17) + P(X=18) + P(X=19) + P(X=20)$$

$$= \frac{\binom{16}{2}}{\binom{20}{3}} + \frac{\binom{17}{2}}{\binom{20}{3}} + \frac{\binom{18}{2}}{\binom{20}{3}} + \frac{\binom{19}{2}}{\binom{20}{3}}$$

$$= .105 + .119 + .134 + .150$$

$$P(X \geq 17) = .508.$$

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2. Let X_i: event that the i th. subsystem will have to be replaced within the first 150 hrs. of operation.

$$P(X_i) = \int_0^{150} f(x)\,dx$$

$$= \int_{100}^{150} \frac{100}{x^2}\,dx$$

$$= 100\int_{100}^{150} x^{-2}\,dx$$

$$= 100\left(\frac{x^{-1}}{-1}\right)\Bigg|_{100}^{150}$$

$$= -100\left[\frac{1}{150} - \frac{1}{100}\right]$$

$$P(X_i) = \frac{1}{3}.$$

Hence, letting R denote the event that exactly 2 of the 5 subsystems must be replaced, and remembering that all 5 subsystems are independent:

$$P(R) = \binom{5}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^3 = \frac{80}{243}.$$

3.

$$F(y) = \int_0^y f(x)\,dx \qquad y \geq 100$$

$$= 100 \int_{100}^{y} x^{-2}\,dx$$

$$= -100 \left(\frac{1}{x}\right)\Big|_{100}^{y}$$

$$= \frac{-100}{y} + 1$$

$$F(y) = 1 - \frac{100}{y}.$$

$\therefore$

$$F(y) = \begin{cases} 0 & \text{if } y \leq 100 \\ 1 - \frac{100}{y} & \text{if } y > 100. \end{cases}$$

A. $P(X \leq 100) = 0 = F(100)$.

B. $P(X \leq 130) = F(130) = 1 - \frac{100}{130} = \frac{30}{130} = \frac{3}{13}$.

C. $P(X > 140) = 1 - F(140) = 1 - \left(1 - \frac{100}{140}\right) = \frac{100}{140} = \frac{5}{7}$

D. $P(110 \leq X \leq 120) = F(120) - F(110)$

$$= \left(1 - \frac{100}{120}\right) - \left(1 - \frac{100}{110}\right) = \frac{5}{66}.$$

4. A. Let A: event passenger waits less than 5 mins. for a bus.

$$P(A) = P(10 < X < 15) + P(25 < X < 30)$$

$$= \int_{10}^{15} \frac{1}{30}\,dx + \int_{25}^{30} \frac{1}{30}\,dx$$

$$= \left.\frac{x}{30}\right|_{10}^{15} + \left.\frac{x}{30}\right|_{25}^{30}$$

$$= \frac{1}{30}(15-10) + \frac{1}{30}(30-25)$$

$$= \frac{5}{30} + \frac{5}{30}$$

$$P(A) = \frac{1}{3}.$$

B. Let B: event passenger waits more than 10 mins. for a bus.

$$P(B) = P(0 < X < 5) + P(15 < X < 20)$$

$$= \int_{0}^{5} \frac{1}{30}\,dx + \int_{15}^{20} \frac{1}{30}\,dx$$

$$= \left.\frac{x}{30}\right|_{0}^{5} + \left.\frac{x}{30}\right|_{15}^{20}$$

$$P(B) = \frac{1}{30}(5) + \frac{1}{30}(5) = \frac{1}{3}.$$

5.A. $p(x, y)$

$x \backslash y$	0	1	2	3
0	$\frac{1}{8}$	$\frac{1}{8}$	0	0
1	0	$\frac{1}{4}$	$\frac{1}{4}$	0
2	0	0	$\frac{1}{8}$	$\frac{1}{8}$

B. $P(Y=1) = \sum_{x=0}^{2} p(x, 1) = \frac{3}{8}$.

C. $P(X=0) = \sum_{y=0}^{3} p(0, y) = \frac{1}{4}$.

6.

Let $X_i = \begin{cases} 1 & \text{if } i\text{th. coin is tails} \\ -1 & \text{" " " " heads.} \end{cases}$

X : T − H. $X = \sum_{i=1}^{N} X_i$.

A. $E[X] = E\left[\sum_{i=1}^{N} X_i\right] = \sum_{i=1}^{N} E[X_i]$.

$$E[X_i] = (1)\left(\frac{1}{2}\right) + (-1)\left(\frac{1}{2}\right) = 0$$

$$\therefore \quad E[X] = \sum_{i=1}^{N} E[X_i] = 0.$$

B. $$Var(X) = Var\left(\sum_{i=1}^{N} X_i\right)$$

$$= \sum_{i=1}^{N} Var(X_i).$$

$$Var(X_i) = E[X_i^2] - \left(E[X_i]\right)^2$$

$$= E[X_i^2] - 0$$

$$= E[X_i^2].$$

$$E[X_i^2] = (1)^2\left(\frac{1}{2}\right) + (-1)^2\left(\frac{1}{2}\right) = 1 \Rightarrow Var(X_i) = 1.$$

$$\therefore \quad Var(X) = \sum_{i=1}^{N} Var(X_i) = \sum_{i=1}^{N} (1) = N.$$

7. Method I.

Let X: number of balls which fall into bin 1.

$$X_i = \begin{cases} 1 & \text{if } i\text{th. ball falls into bin 1} \\ 0 & \text{otherwise.} \end{cases}$$

$\therefore, \; X = \sum_{i=1}^{M} X_i.$

$$E[X] = E\left[\sum_{i=1}^{M} X_i\right] = \sum_{i=1}^{M} E[X_i].$$

$$E[X_i] = (1)\left(\frac{1}{N}\right) + (0)\left(\frac{N-1}{N}\right) = \frac{1}{N}.$$

$\left(\text{NOTA BENE: } P(X_i = 1) = \frac{1}{N}\right).$

$\therefore,$

$$E[X] = \sum_{i=1}^{M} E[X_i] = \sum_{i=1}^{M}\left(\frac{1}{N}\right) = \frac{M}{N}.$$

METHOD 2.

Probability of getting a ball randomly into a bin is $\frac{1}{N}$.

There are M balls.

$\therefore,$

$$E[X] = (M)\left(\frac{1}{N}\right) = \frac{M}{N}.$$

8. A.

$$\text{P.D.F.: } f(x) = \begin{cases} \frac{1}{5} & 0 < x \leq 5 \\ 0 & \text{elsewhere.} \end{cases}$$

$\left(\text{NOTA BENE: } f(x) = F'(x)\right).$

B. $E[X] = \int_0^5 x\, f(x)\, dx$

$= \int_0^5 \frac{x}{5}\, dx$

$= \frac{x^2}{10}\Big|_0^5$

$E[X] = \frac{5}{2}.$

C. $E[5-X] = \int_0^5 (5-x)\, f(x)\, dx$

$= \int_0^5 (5-x)\left(\frac{1}{5}\right) dx$

$= \left(x - \frac{x^2}{10}\right)\Big|_0^5$

$= 5 - \frac{5}{2}$

$E[5-X] = \frac{5}{2}.$

D. $E[X(5-X)] = \int_0^5 x(5-x)\, f(x)\, dx$

$= \frac{1}{5}\int_0^5 x(5-x)\, dx$

$$= \frac{1}{5}\int_0^5 (5x - x^2)\,dx$$

$$= \frac{1}{5}\left(\frac{5x^2}{2} - \frac{x^3}{3}\right)\Bigg|_0^5$$

$$= \frac{1}{5}\left(\frac{125}{2} - \frac{125}{3}\right)$$

$$= \frac{25}{2} - \frac{25}{3}$$

$$E[X(5-X)] = \frac{25}{6} \neq \left(\frac{5}{2}\right)^2 = E[X]\,E[5-X].$$

$\therefore$ NO!

9. A.

Let $\phi_X(t)$: Moment Generating Function of X.

$\phi_{X_1}(t)$: " " " " X_1.

$\phi_{X_2}(t)$: " " " " X_2.

$$\phi_X(t) = E\left[e^{tX}\right] = E\left[e^{t(X_1+X_2)}\right]$$

$$= E\left[e^{tX_1} e^{tX_2}\right]$$

$$= E\left[e^{tX_1}\right] E\left[e^{tX_2}\right]$$

$$\phi_X(t) = \phi_{X_1}(t)\, \phi_{X_2}(t).$$

$$\phi_{X_1}(t) = \phi_{X_2}(t).$$

$$\phi_{X_1}(t) = E\left[e^{tX_1}\right] = \sum_{i=1}^{3} e^{tx} p(x)$$

$$= \frac{1}{6} e^{t} + \frac{2}{6} e^{2t} + \frac{3}{6} e^{3t}.$$

$$\therefore)\quad \phi_X(t) = \left(\frac{1}{6} e^{t} + \frac{2}{6} e^{2t} + \frac{3}{6} e^{3t}\right)^2$$

$$= \frac{1}{36} e^{2t} + \frac{4}{36} e^{3t} + \frac{10}{36} e^{4t} + \frac{12}{36} e^{5t} + \frac{9}{36} e^{6t}.$$

B.

$$E[X] = \phi_X'(0) = \frac{2}{36} + \frac{12}{36} + \frac{40}{36} + \frac{60}{36} + \frac{54}{36}$$

$$= \frac{168}{36}$$

$$E[X] = \frac{14}{3}.$$

C.

$$E[X^2] = \phi_X''(0)$$

$$= \frac{4}{36} + \frac{36}{36} + \frac{160}{36} + \frac{300}{36} + \frac{324}{36}$$

$$= \frac{824}{36}$$

$$E[X^2] = \frac{206}{9}.$$

$$\therefore Var(X) = E[X^2] - (E[X])^2$$

$$= \frac{206}{9} - \frac{196}{9}$$

$$Var(X) = \frac{10}{9}.$$

10. Use Chebyshev's Inequality.

$$P(|X - \mu| \geq K\sigma) \leq \frac{1}{K^2}$$

$\mu = 0;\ \sigma^2 = 4 \Rightarrow \sigma = 2.\ \forall x \in X,\ x > 0.$

Let $K = \frac{3}{2}$.

$$\therefore P(|X - \mu| \geq K\sigma) \leq \frac{1}{K^2}$$

$$P\left(|X - 0| \geq \left(\frac{3}{2}\right)(2)\right) \leq \frac{1}{\left(\frac{3}{2}\right)^2}$$

$$P(|X| \geqslant 3) \leq \frac{4}{9}$$

$$P(X \geqslant 3) \leq \frac{4}{9}.$$

COMP. 233.

assign. #2 Solutions. Problems 11-17.

11. A.

$$p(x) = \sum_{y=1}^{\infty} p(x,y) = \sum_{y=1}^{\infty} 2^{-x-y}$$

$$= \sum_{y=1}^{\infty} 2^{-x} 2^{-y}$$

$$= 2^{-x} \sum_{y=1}^{\infty} 2^{-y}$$

$$= 2^{-x}(1)$$

$$p(x) = 2^{-x}.$$

Similarly for $p(y)$; $p(y) = 2^{-y}$!

B.

$p(x,y) = 2^{-x-y} = 2^{-x} 2^{-y} = p(x)\,p(y) \Rightarrow$

$p(x,y) = p(x)\,p(y) \Rightarrow$ R.V.'s X, Y are independent.

C.

$$E[X] = \sum_{x=1}^{\infty} x\, p(x) = \sum_{x=1}^{\infty} x\, 2^{-x} = \sum_{x=1}^{\infty} \frac{x}{2^x} = 2.$$

Similarly $E[Y] = 2$.

NOTA BENE:

$$\frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + 4x^4 + \ldots = \sum_{n=1}^{\infty} n x^n.$$

Let $x = \frac{1}{2}$;

$$2 = \frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \frac{4}{2^4} + \ldots = \sum_{n=1}^{\infty} n\left(\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} \frac{n}{2^n}.$$

$$E[XY] = \sum_{x=1}^{\infty}\sum_{y=1}^{\infty} xy\, P(x,y) = \sum_{x=1}^{\infty}\sum_{y=1}^{\infty} xy\, 2^{-x-y}$$

$$= \sum_{x=1}^{\infty} x 2^{-x} \sum_{y=1}^{\infty} y 2^{-y}$$

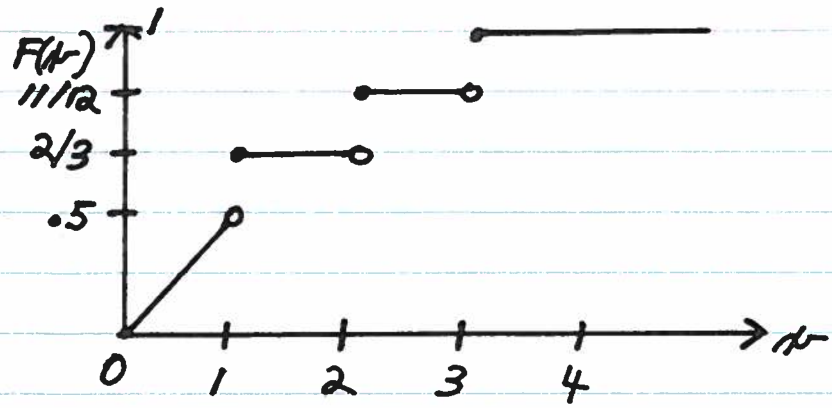
$$E[XY] = (2)(2) = 4.$$

OR

Since X and Y are independent $\Rightarrow$

$$E[XY] = E[X]E[Y] = (2)(2) = 4.$$

12. A.



B. $P(X > .5) = 1 - P(X \leq .5) = 1 - F(.5)$
$= 1 - \frac{1}{4} = \frac{3}{4}.$

C. $P(2 < X \leq 4) = P(X \leq 4) - P(X \leq 2)$
$= F(4) - F(2)$
$= 1 - \frac{11}{12} = \frac{1}{12}.$

D. $P(X < 3) = \frac{11}{12}.$

E. $P(X=1) = P(X \leq 1) - P(X < 1) = F(1) - P(X < 1)$
$= \frac{2}{3} - \frac{1}{2} = \frac{1}{6}.$

13. A.

$$f(x) = \int_0^1 (x+y)\,dy$$

$$= \left(xy + \frac{y^2}{2}\right)\Big|_0^1$$

$$= \left(x + \frac{1}{2}\right) - (0)$$

$$f(x) = x + \frac{1}{2}.$$

$\therefore$,

$$f(x) = \begin{cases} x + \frac{1}{2} & 0 \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

Similarly for $f(y)$:

$$f(y) = \begin{cases} y + \frac{1}{2} & 0 \le y \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

B. $E[X] = \int_0^1 x\,f(x)\,dx = \int_0^1 x(x + .5)\,dx$

$$= \int_0^1 (x^2 + .5x)\,dx$$

$$= \left(\frac{x^3}{3} + \frac{x^2}{4}\right)\Big|_0^1$$

$$E[X] = \left(\frac{1}{3} + \frac{1}{4}\right) - (0) = \frac{7}{12}.$$

$$E[X^2] = \int_0^1 x^2 f(x)\,dx$$

$$= \int_0^1 x^2 (x + .5)\,dx$$

$$= \int_0^1 (x^3 + .5x^2)\,dx$$

$$= \left(\frac{x^4}{4} + \frac{x^3}{6}\right)\Big|_0^1$$

$$= \left(\frac{1}{4} + \frac{1}{6}\right) + (0)$$

$$E[X^2] = \frac{5}{12}.$$

$$\therefore\ VAR(Y) = VAR(X) = E[X^2] - (E[X])^2$$

$$= \frac{5}{12} - \left(\frac{7}{12}\right)^2$$

$$= \frac{5}{12} - \frac{49}{144}$$

$$VAR(Y) = \frac{11}{144}.$$

14. A. $\int_{-\infty}^{\infty} f(z)\,dz = 1$

$$\int_0^1 kz^2\,dz + \int_1^{4/3} (1)\,dz = 1$$

$$\frac{kz^3}{3}\Big|_0^1 + z\Big|_1^{4/3} = 1$$

$$\frac{k}{3} + \left(\frac{4}{3} - 1\right) = 1$$

$$\frac{k}{3} + \frac{1}{3} = 1$$

$$\frac{k}{3} = \frac{2}{3}$$

$$k = 2.$$

B. $P(0 \le Z \le 1) = \int_0^1 2z^2\,dz = \frac{2}{3}z^3\Big|_0^1 = \frac{2}{3}.$

C. $E[Z] = \int_{-\infty}^{\infty} z f(z)\,dz$

$$= \int_0^1 2z^3\,dz + \int_1^{4/3} z\,dz$$

$$= \frac{z^4}{2}\Big|_0^1 + \frac{z^2}{2}\Big|_1^{4/3}$$

$$E[Z] = \frac{1}{2} + \left(\frac{8}{9} - \frac{1}{2}\right) = \frac{8}{9}.$$

$$E[Z^2] = \int_{-\infty}^{\infty} z^2 f(z)\,dz$$

$$= \int_0^1 2z^4\,dz + \int_1^{4/3} z^2\,dz$$

$$= \frac{2}{5}z^5\Big|_0^1 + \frac{z^3}{3}\Big]_1^{4/3}$$

$$= \frac{2}{5} + \left(\frac{64}{81} - \frac{1}{3}\right)$$

$$E[Z^2] = \frac{2}{5} + \frac{37}{81} = \frac{347}{405}.$$

$$\therefore\ Var(Z) = E[Z^2] - (E[Z])^2$$

$$= \frac{347}{405} - \left(\frac{8}{9}\right)^2$$

$$= \frac{347}{405} - \frac{64}{81}$$

$$= \frac{27}{405}$$

$$Var(Z) = \frac{1}{15}.$$

15. A. Markov's Inequality.

$$P(X \geq 75) \leq \frac{E[X]}{75} = \frac{65}{75} = \frac{13}{15}.$$

B. Chebyshev's Inequality.

$$P(|X-\mu| \geq k) \leq \frac{\sigma^2}{k^2}$$

$$P(|X-65| \geq 10) \leq \frac{30}{10^2} = \frac{30}{100} = .3.$$

$$1 - P(|X-65| \geq 10) \geq 1-.3$$

$$P(|X-65| < 10) \geq .7$$

$$P(-10 < X-65 < 10) \geq .7$$

$$P(55 < X < 75) \geq .7.$$

C. Class average: $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$.

$$Var(\bar{X}) = Var\left(\frac{X_1}{n} + \frac{X_2}{n} + \frac{X_3}{n} + \dots + \frac{X_n}{n}\right)$$

$$= Var\left(\frac{X_1}{n}\right) + Var\left(\frac{X_2}{n}\right) + Var\left(\frac{X_3}{n}\right) + \dots + Var\left(\frac{X_n}{n}\right)$$

$$= \frac{1}{n^2}Var(X_1) + \frac{1}{n^2}Var(X_2) + \dots + \frac{1}{n^2}Var(X_n)$$

$$= \frac{30}{n^2} + \frac{30}{n^2} + \frac{30}{n^2} + \frac{30}{n^2} + \cdots\cdots + \frac{30}{n^2}$$

$$= \frac{30n}{n^2}$$

$$Var(\bar{X}) = \frac{30}{n}.$$

Chebyshev's Inequality.

$$P(|\bar{X} - \mu| \geq k) \leq \frac{Var(\bar{X})}{k^2}$$

$$P(|\bar{X} - 65| \geq 5) \leq \frac{\left(\frac{30}{n}\right)}{25}$$

$$P(|\bar{X} - 65| \geq 5) \leq \frac{6}{5n}.$$

$$1 - P(|\bar{X} - 65| \geq 5) \geq 1 - \frac{6}{5n}$$

$$P(|\bar{X} - 65| < 5) \geq 1 - \frac{6}{5n}$$

$$P(-5 < \bar{X} - 65 < 5) \geq 1 - \frac{6}{5n}$$

$$P(60 < \bar{X} < 70) \geq 1 - \frac{6}{5n}.$$

Since this probability must be $\geq .8 \Rightarrow$

$$1-\frac{6}{5n}=.8=\frac{8}{10}$$

$$\frac{6}{5n}=\frac{2}{10}$$

$$60=10n$$

$n=6 \Rightarrow$ at least 6 students are neccessary.

16. A. message incorrectly decoded.

If a "0" is to be received ("00000") then an incorrect decoding would happen if at least 3 "1" are decoded.

Hence,

$$P(\text{Incorrect Decoding}) = P(X \geq 3)$$

where X: number of "1" decoded.

$$P(X \geq 3) = \sum_{x=3}^{5} B(x;5,.2) = .0579.$$

B. The assumption made is that each digit is transmitted independently of the others.

17. A. Prob. a couple is born on April 1st. is:

$$\left(\frac{1}{365}\right)\left(\frac{1}{365}\right) = \frac{1}{(365)^2}.$$

Prob. at least one couple is born on April 1st is:

$$\sum_{i=1}^{20,000} B(i; 20,000, 1/(365)^2)$$

$$= 1 - B\left(0; 20,000, \frac{1}{(365)^2}\right)$$

$$= 1 - \binom{20,000}{0}\left(\frac{1}{(365)^2}\right)^0\left(1 - \frac{1}{(365)^2}\right)^{20,000} \approx .139.$$

B. Prob. a couple is born on a specific day is: $\frac{1}{(365)^2}$.

Prob. a couple is born on any day of the year is:

$$365\left(\frac{1}{(365)^2}\right) = \frac{365}{(365)^2} = \frac{1}{365}.$$

Prob. at least one couple born on same day is:

$$\sum_{i=1}^{20,000} B(i; 20,000, 1/365)$$

$$= 1 - B(0; 20,000, 1/365) = 1 - \binom{20,000}{0}\left(\frac{1}{365}\right)^0\left(\frac{364}{365}\right)^{20000}$$

$$\approx 1.$$