$Solve\ the\ differential\ equation$

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad where\ \ u(0,t) = 0, u(1,t) = 0,\ \ u(x,0) = 1 \qquad [7]$$

Solution

$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}\ \ homogeneous \qquad 0 < x < 1,\ \ t \geq 0$

$u(0,t) = 0,\ \ u(1,t) = 0, \qquad both\ are\ homogeneous\ \ u(x,0) = 1\ \ nonhomogeneous$

$u(x,t) = T(t)X(x) \quad \frac{T_t}{T} = \frac{X_{xx}}{X} = -\lambda \qquad u(0,t) = 0,\ \ u(1,t) = 0,$

$$X_{xx} + \lambda X = 0, \qquad X(0) = 0, \quad X(1) = 0 \qquad Eigenvalues\ \ \lambda_n = (n\pi)^2 \quad n = 1,2,3 \ldots$$

$$X_n(x) = c_n sin(n\pi x)$$

$$The\ solution\ of\ T_t + \lambda T = 0\ is\ T(t) = a_1 e^{-\lambda t}, \qquad T_n = a_n e^{-n^2\pi^2 t}$$

$u(x,t) = \sum_{n-1}^{\infty} k_n e^{-n^2\pi^2 t} sin n\pi x \qquad u(x,0) = \sum_{n-1}^{\infty} k_n sin n\pi x = 1$

$$k_n = \frac{2}{p}\int_0^p f(x)\, sin\frac{n\pi x}{p} dx$$

$$k_n = 2\int_0^1 1.\ sin n\pi x dx = -\frac{2}{n\pi}\{cos n\pi - 1\}$$

$$u(x,t) = \frac{2}{n\pi}\sum_{n-1}^{\infty}\{1 - cos n\pi\} e^{-n^2\pi^2 t} sin n\pi x$$