MAT1332: Calculus for the Life Sciences II

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1 Review of integrals

1.1 Power rule

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c \text{ if } n \neq -1$$

Example 1.

$$\int t^3 dt = \frac{t^4}{4} + c$$

Example 2.

$$\int y^{-3} dy = \frac{y^{-2}}{-2} + c$$

1.2 Special Functions

Exponentials:

$$\int e^x dx = e^x + c$$

Logarithms:

$$\int \frac{1}{x} dx = \ln|x| + c$$

Trigonometric functions:

$$\int \sin x dx = -\cos x + c$$
$$\int \cos x dx = \sin x + c$$

Example 3.

$$\int \frac{1}{1+y^2} dy = \arctan y + c$$

Hint:

$$\arctan(\tan x) = x$$

Let $y = \tan x$. Then, differentiating both sides with respect to x, we have

$$
\begin{aligned}
\frac{d}{dx}\arctan y &= \frac{d}{dx}x \\
\text{Chain rule}: \frac{d}{dy}\arctan y \cdot \frac{d}{dx}\tan x &= 1 \\
\frac{d}{dy}\arctan y \cdot \frac{d}{dx}\frac{\sin x}{\cos x} &= 1 \\
\frac{d}{dy}\arctan y \cdot \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{\cos^2(x)} &= 1 \\
\frac{d}{dy}\arctan y \cdot \frac{\cos^2 x + \sin^2 x}{\cos^2 x} &= 1 \\
\frac{d}{dy}\arctan y \cdot (1 + \tan^2 x) &= 1 \\
\frac{d}{dy}\arctan y \cdot (1 + y^2) &= 1 \\
\frac{d}{dy}\arctan y &= \frac{1}{1+y^2} \\
\therefore \int \frac{1}{1+y^2}dy &= \arctan y + c
\end{aligned}
$$

1.3 Substitution

$$
\begin{aligned}
\frac{d}{dx}f(g(x)) &= f'(g(x))g'(x) \\
f(g(x)) &= \int f'(g(x))g'(x)dx \\
\textit{substitute } u &= g(x) \\
\frac{du}{dx} &= g'(x) \\
dx &= \frac{du}{g'(x)} \\
\textit{then } f(g(x)) &= \int f'(u)g'(x)\frac{du}{g'(x)} \\
&= \int f'(u)du
\end{aligned}
$$

Example 4. $\int 3x^2 \sin x^3 dx$

$$
\begin{aligned}
\int 3x^2 \sin x^3 dx &= \int 3x^2 \sin u \cdot \frac{du}{3x^2} & u &= x^3 \\
&= \int \sin u \cdot du & \frac{du}{dx} &= 3x^2 \\
&= -\cos u + c & dx &= \frac{du}{3x^2} \\
&= -\cos x^3 + c
\end{aligned}
$$

1.4 Integration by Parts

$$\int uv' = uv - \int u'v$$

Example 5. $\int x \cdot \cos x dx$

$$\begin{aligned} u &= x & v' &= \cos x \\ u' &= 1 & v &= \sin x \end{aligned}$$

$$\begin{aligned} \int x \cdot \cos x dx &= x \sin x - \int \sin x dx \\ &= x \cdot \sin x + \cos x + c \end{aligned}$$

1.5 Important properties

Integrals preserve sums: $\int (f(x) + g(x))dx = \int f(x)dx + \int g(x)dx$

Example 6.

$$\int (\frac{1}{x^2} + \frac{1}{x})dx$$

$$\begin{aligned} \int (\frac{1}{x^2} + \frac{1}{x})dx &= \int (x^{-2} + \frac{1}{x})dx \\ &= \int x^{-2}dx + \int \frac{1}{x}dx \\ &= \frac{x^{-1}}{-1} + \ln|x| + c \\ &= -\frac{1}{x} + \ln|x| + c \end{aligned}$$

Integrals preserve constant products: $\int af(x)dx = a \int f(x)dx$

Example 7.

$$\int 6x^{-1}dx = 6 \int x^{-1}dx = 6 \ln|x| + c$$

Example 8. $\int 3xe^{x^2}dx$

Try moving the constant outside: $\int 3xe^{x^2}dx = 3 \int xe^{x^2}dx$, which is not very helpful.

Try integration by parts:

$$\begin{aligned} u &= 3x & v' &= e^{x^2} \\ u' &= 3 & v &= ?? \end{aligned}$$

Or

$$\begin{aligned} u &= e^{x^2} & v' &= 3x \\ u' &= 2xe^{x^2} & v &= \frac{3}{2}x^2 \end{aligned}$$

Then $\int 3xe^{x^2}dx = \frac{3}{2}x^2e^{x^2} - \int 3x^3e^{x^2}dx$ which is more complicated.

Try substitution:

$$\begin{aligned}
&Try\ u = e^{x^2}? \\
&\qquad \frac{du}{dx} = 2xe^{x^2} \text{does not obviously cancel (though this actually works)} \\
&Try\ u = 3x? \\
&\qquad \frac{du}{dx} = 3 \\
&\qquad \int 3xe^{x^2}\frac{du}{3} = \int \frac{u}{3}e^{\frac{u^2}{9}}du \text{ is not significantly different} \\
&Try\ u = x^2? \\
&\qquad \frac{du}{dx} = 2x \\
&\int 3xe^u\frac{du}{2x} = \int \frac{3}{2}e^u du \\
&\qquad = \frac{3}{2}\int e^u du \\
&\qquad = \frac{3}{2}e^u + c \\
&\qquad = \frac{3}{2}e^{x^2} + c
\end{aligned}$$

Example 9. Solve $\int (5x^4 - 2x^3 + 3)dx$

$$\int (5x^4 - 2x^3 + 3)dx = \frac{5x^5}{5} - \frac{2x^4}{4} + 3x + c = x^5 - \frac{x^4}{2} + 3x + c$$

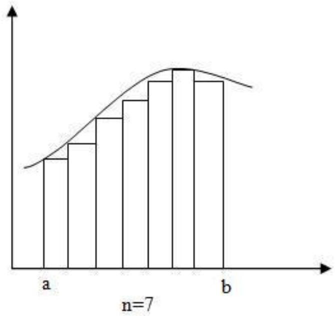
What are some applications of integrals?

- Averages
- Probabilities
- Areas
- Sums

Exercise: $\int \frac{(t+3)^2}{t}dt$
Exercise: $\int x^3e^x dx$
Exercise: $\int \ln x dx$
Exercise: $\int x \ln x dx$
Exercise: $\int x \sin x \cos x dx$
Exercise: $\int e^x \cos x dx$

2 Integrals

Definition 2.1. *The integral is defined as* $\int_a^b f(t)dt = \lim_{n\to\infty}\sum_{i=1}^{n-1} f(t_i)\Delta t$ *where the values* $t_0, ..., t_n$ *break the interval from* a *to* b *into* n *pieces, each of width* $\Delta t = \frac{b-a}{n}$.

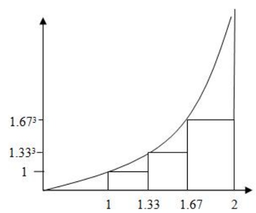


We call these Reimann Sums.

2.1 Riemann Sums

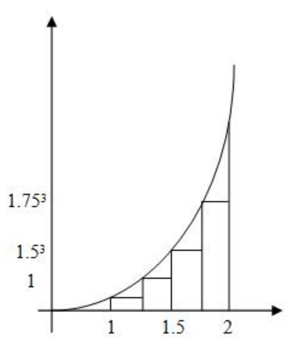
Example 10. Evaluate $\int_1^2 x^3 dx$ using 3, 4 and 10 Riemann sums.

$$Area \approxeq \frac{1}{3}(1)^3 + \frac{1}{3}(1.33)^3 + \frac{1}{3}(1.67)^3 = \frac{8}{3} = 2.667$$



For 4 sums, we have:

$$Area \approxeq \frac{1}{4}(1)^3 + \frac{1}{4}(1.25)^3 + \frac{1}{4}(1.5)^3 + \frac{1}{4}(1.75)^3 = 2.922$$



For 10 sums, we have:

$$Area \approxeq 0.1(1^3 + 1.1^3 + 1.2^3 + 1.3^3 + 1.4^3 + 1.5^3 + 1.6^3 + 1.7^3 + 1.8^3 + 1.9^3) = 3.4075$$

3 Definite and Indefinite Integrals

Theorem 3.1. *(The Fundamental Theorem of Calculus) Suppose* $\frac{dF}{dt} = f(t)$*. The indefinite integral is* $\int f(t)dt = F(t) + c$*.*

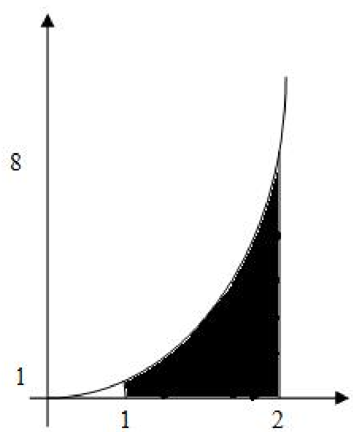
The definite integral is

$$\int_a^b f(t)dt = [F(t) + c]_{evaluated\ at\ b} - [F(t) + c]_{evaluated\ at\ a}$$
$$= (F(b) + c) - (F(a) + c) = F(b) - F(a).$$

Example 11.

$$\int_1^2 x^3 dx$$

$$\begin{aligned}\int_1^2 x^3 dx &= \left.\frac{x^4}{4}\right|_1^2 \\ &= \frac{2^4}{4} - \frac{1^4}{4} \\ &= \frac{16}{4} - \frac{1}{4} \\ &= 4 - \frac{1}{4} \\ &= 3\frac{3}{4} \\ &= \frac{15}{4} \\ &= 3.75\end{aligned}$$

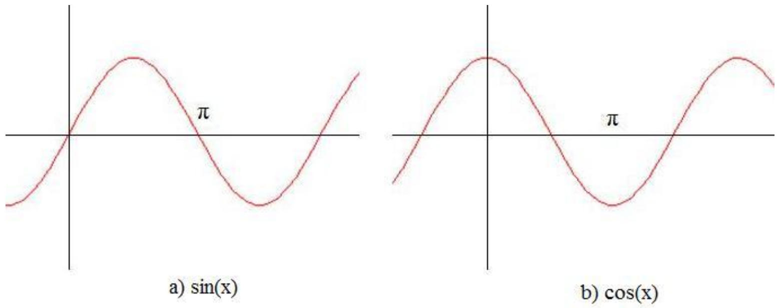


Example 12.

$$\int_0^\pi \sin x dx$$

$$\begin{aligned}\int_0^\pi \sin x dx &= \big[-\cos x\big]_0^\pi \\ &= (-\cos\pi) - (-\cos 0) \\ &= 1 + 1 \\ &= 2\end{aligned}$$

Exercise: Show that $\cos x$ and $\sin x$ have the same area for one whole period.



Example 13. A rock is hurled down from a building 100 m high with inital speed of 5 m/s. How far will it fall during the first second?

Facts: $a = \frac{dv}{dt} = -9.8m/s^2$ $\quad v = \frac{dp}{dt}$

$$\begin{aligned} v &= \int a dt \\ &= \int -9.8 dt \\ &= -9.8t + v_0 \end{aligned}$$

$$\begin{aligned} \text{We know that } v(0) = -9.8(0) + v_0 = -5 &\to v_0 = -9.8 - 5 \\ v &= -9.8t - 5 \\ p &= \int_0^1 v dt \\ &= \int_0^1 (-9.8t - 5) dt \\ &= \left[\frac{-9.8t^2}{2} - 5t \right]_0^1 \\ &= \left[\frac{-9.8}{2} - 5 \right] - [0 - 0] \\ &= -4.9 - 5 \\ &= -9.9 \end{aligned}$$

Therefore it falls down 9.9 metres in the first second.

Exercise: How far will it fall in 4 seconds? How far in 5 seconds? (Be careful)

Example 14. A fish grows at rate $\frac{dL}{dt} = 3e^{-0.5t}$ where t is time in years and L is length in centimetres. How much does it grow between the ages of 3 and 6?

$$\text{We must find } \int_3^6 3e^{-0.5t}dt$$

$$\text{Substitute : } u = -0.5t \qquad \frac{du}{dt} = -0.5 \qquad dt = -2du$$

$$\begin{aligned}
\int_3^6 3e^{-0.5t}dt &= \int_{t=3}^{t=6} 3e^{u}(-2)du \\
&= -6\int_{t=3}^{t=6} e^{u}du \\
&= -6e^{u}\Big|_{t=3}^{t=6} \\
&= -6e^{-0.5t}\Big|_{t=3}^{t=6} \\
&= -6e^{-0.5(6)} - (-6e^{-0.5(3)}) \\
&= -6e^{-3} + 6e^{-\frac{3}{2}} \\
&= 1.04\ cm
\end{aligned}$$

Exercise: How much does it grow before age 3?

Example 15. The rate at which you learn math is $\frac{dC}{dt} = 50t^2e^{-t}$ where C is a measure of comprehension and t is time in weeks. How much will you learn in 1 week, 6 weeks, 13 weeks?

General solution:

$$C = 50\int t^2e^{-t}dt$$

$$\textit{Let } u = t^2 \qquad v' = e^{-t}$$

$$u' = 2t \qquad v = -e^{-t}$$

$$C = 50[-t^2e^{-t} + 2\int te^{-t}dt]$$

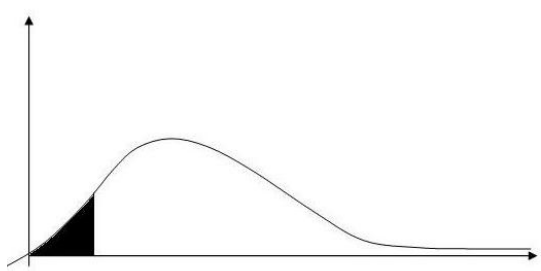
$$\textit{Let } u = 2t \qquad v' = e^{-t}$$

$$u' = 2 \qquad v = -e^{-t}$$

$$\begin{aligned}
C &= 50[-t^2e^{-t} - 2te^{-t} + 2\int e^{-t}dt] \\
&= 50[-t^2e^{-t} - 2te^{-t} - 2e^{-t}]
\end{aligned}$$

In one week:

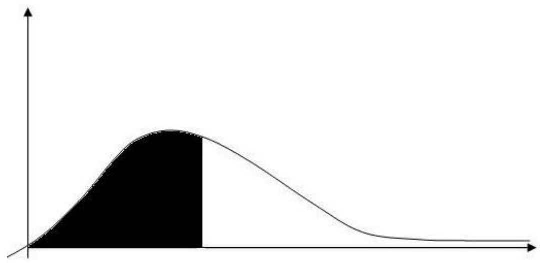
$$\begin{aligned}
C &= 50[-t^2e^{-t} - 2te^{-t} - 2e^{-t}]_0^1 \\
&= 50[(-e^{-1} - 2e^{-1} - 2e^{-1}) - (0 - 0 - 2)] \\
&= 50[2 - 5e^{-1}] \\
&= 100 - 250e^{-1}
\end{aligned}$$



$$= 8.03\%$$

In 6 weeks:

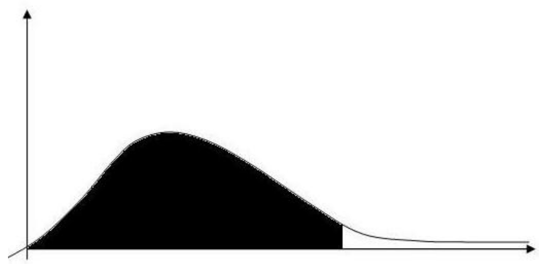
$$\begin{aligned} C &= 50[-t^2e^{-t} - 2te^{-t} - 2e^{-t}]_0^6 \\ &= 50[(-36e^{-6} - 12e^{-6} - 2e^{-6}) - (0 - 0 - 2)] \\ &= 50[-50e^{-6} + 2] \end{aligned}$$



$= 93.8\%$

In 13 weeks:

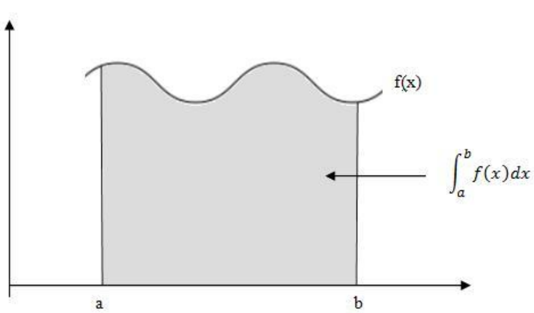
$$\begin{aligned} C &= 50[-t^2e^{-t} - 2te^{-t} - 2e^{-t}]_0^{13} \\ &= 50[(-169e^{-13} - 26e^{-13} - 2e^{-13}) - (0 - 0 - 2)] \\ &= 50[2 - 197e^{-13}] \end{aligned}$$



$= 99.98\%$

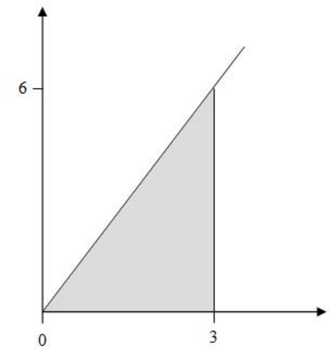
4 Applications of integrals

Definition 4.1. *A definite integral is an area under the curve between the limits.*



Example 16. Find the area under the line $f(x) = 2x$ between 0 and 3.

$$
\begin{aligned}
Area &= \int_0^3 2x dx \\
&= \left[\frac{2x^2}{2}\right]_0^3
\end{aligned}
$$



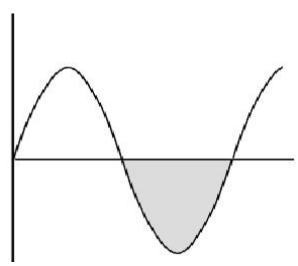
$$= 9 \; units^2$$

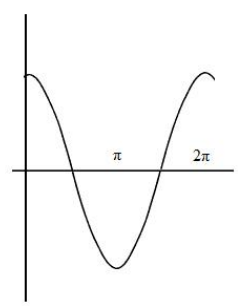
But this area is a triangle with base=3 and height=6:

$$\therefore \; Area = \frac{b \cdot h}{2} = \frac{3 \cdot 6}{2} = 9$$

Example 17. Find the area under the curve $f(x) = \sin x$ between π and 2π.

$$
\begin{aligned}
Area &= \int_\pi^{2\pi} \sin x dx \\
&= [-\cos x]_\pi^{2\pi} \\
&= -\cos 2\pi - (-\cos \pi) \\
&= -1 - (-(-1))
\end{aligned}
$$

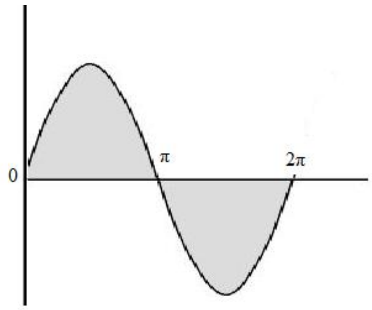




$$= -2$$

How can this be? The definite integral gives a positive area if the curve is above the x-axis. The answer will be negative if it is below.

Example 18. Find the total shaded area for one period of $\sin x$.



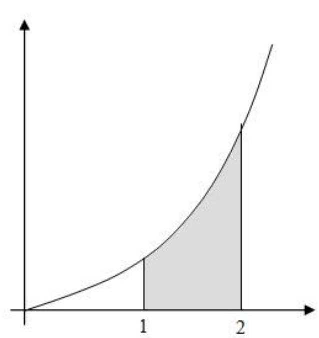
$$\begin{aligned}\int_0^{2\pi} \sin x dx &= [-\cos x]_0^{2\pi} \\ &= -\cos 2\pi - (-\cos 0) \\ &= -1 - (-1) \\ &= 0 \qquad \text{which is clearly not the right answer.}\end{aligned}$$

Try again using absolute value:

$$\begin{aligned}\int_0^{2\pi} |\sin x| dx &= \int_0^{\pi} |\sin x| dx + \int_{\pi}^{2\pi} |\sin x| dx \\ &= \int_0^{\pi} \sin x dx + \int_{\pi}^{2\pi} -\sin x dx \\ &= [-\cos x]_0^{\pi} + [\cos x]_{\pi}^{2\pi} \\ &= [-\cos \pi - (-\cos 0)] + [\cos 2\pi - \cos \pi] \\ &= -(-1) - (-1) + 1 - (-1) \\ &= 4 \; units^2\end{aligned}$$

Example 19. Find $\int_2^1 4x^3 dx$.

$$\begin{aligned}\int_2^1 4x^3 dx &= [x^4]\,_2^1 \\ &= 1^4 - 2^4 \\ &= 1 - 16\end{aligned}$$



$$= -15$$

The answer will be negative if the limits are in the wrong order; that is, the area is positive if the curve is above the area and the limits go from left to right.

In particular

$$\int_b^a f(t)dt = -\int_a^b f(t)dt$$

Example 20. $\int_0^1 (5-2t-t^2)(1+t)dt$. We can try three methods: Substitution, multiplying out and integration by parts.

Try Substitution:

$$\begin{aligned} u &= 5-2t-t^2 \\ \frac{du}{dt} &= -2-2t \\ dt &= \frac{du}{-2(1+t)} \end{aligned}$$

Method 1: Hold limits until the end

$$\begin{aligned} \int_0^1 (5-2t-t^2)(1+t)dt &= \int_{t=0}^{t=1} (u)(1+t)\frac{du}{-2(1+t)} \\ &= -\frac{1}{2}\int_{t=0}^{t=1} udu \\ &= -\frac{1}{2}\left[\frac{u^2}{2}\right]_{t=0}^{t=1} \quad \leftarrow \text{Do not substitute!} \\ &= -\frac{1}{2}\left[\frac{(5-2t-t^2)^2}{2}\right]_{t=0}^{t=1} \\ &= -\frac{1}{4}[(5-2-1)^2-(5-0-0)^2] \\ &= -\frac{1}{4}(2^2-5^2) \\ &= -\frac{1}{4}(4-25) \\ &= -\frac{1}{4}(-21) \\ &= \frac{21}{4} \end{aligned}$$

Method 2: Change limits for new variable

$$\begin{aligned} \text{From the substitution we find } t=0 &\rightarrow u = 5-0-0=5 \\ t=1 &\rightarrow u=5-2-1=2 \\ \int_0^1 (5-2t-t^2)(1+t)dt &= \int_{u=5}^{u=2} u(1+t)\frac{du}{-2(1+t)} \\ &= -\frac{1}{2}\int_5^2 udu \\ &= -\frac{1}{2}\left[\frac{u^2}{2}\right]_5^2 \\ &= -\frac{1}{2}\left[\frac{2^2}{2}-\frac{5^2}{2}\right] \\ &= -\frac{1}{2}\left[\frac{4-25}{2}\right] \\ &= \frac{21}{4} \end{aligned}$$

Either of these methods are fine, but you have to do one of them. Do **not** put the original limits in the answer for the new variable.

Exercise: Try multiplying out and integration by parts.

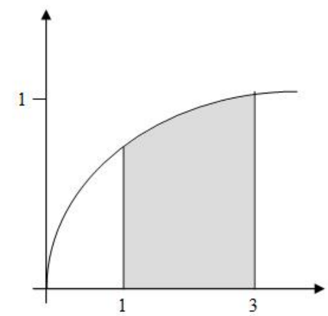
4.1 Integration and Averages

Recall that a rate is an amount per time. That is, *average rate* $= \frac{\textit{total amount}}{\textit{total time}}$.

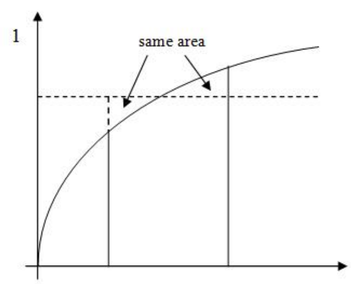
Example 21. Water flows into a vessel at a rate of $1 - e^{-t}$ cm^3/s. What is the average rate at which water enters between $t = 1$ and $t = 3$?

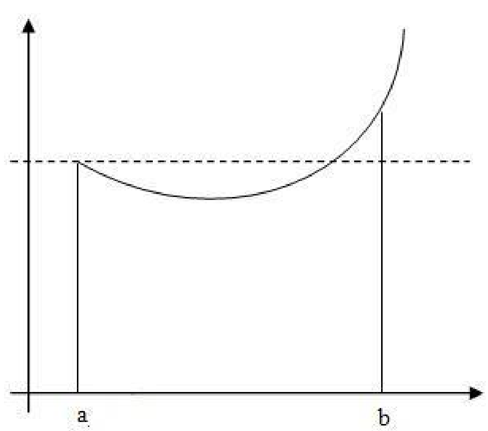
$$\begin{aligned} \textit{Total water entering} &= \int_1^3 (1 - e^{-t})dt \\ &= \left[t - \frac{e^{-t}}{-1}\right]_1^3 \\ &= \left[t + e^{-t}\right]_1^3 \\ &= (3 + e^{-3}) - (1 + e^{-1}) \\ &= 1.682 \end{aligned}$$

$$\textit{Total time} = 3 - 1 = 2$$



$$\therefore \textit{ average rate} = \frac{1.682}{2} = 0.841\ cm^3/s.$$





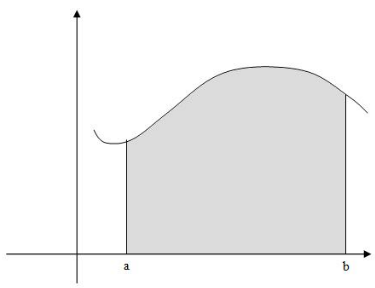
In general, the average value of $f = \frac{1}{b-a}\int_a^b f(x)dx$. The area under the average (between a and b) is equal to the area under the curve (between a and b).

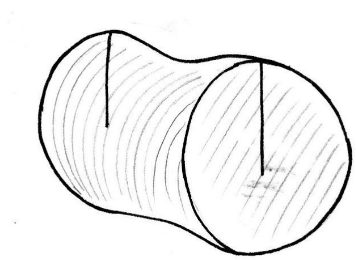
Example 22. In its first decade, the number of AIDS cases in the U.S. followed the formula $\frac{dA}{dt} = 523.8t^2$ where t is the time in years. How many people, on average, were infected each day during this decade?

$$\begin{aligned}
\overline{A} &= \frac{1}{10-0}\int_0^{10} 523.8t^2 dt \\
&= \frac{1}{10}523.8\frac{t^3}{3}\bigg|_0^{10} \\
&= \frac{1}{10}523.8\frac{(10^3)}{3} \\
&= 17,460 \; per \; year \\
Average &= \frac{17,460}{365} = 47.8 \; per \; day
\end{aligned}$$

5 Volumes of Revolution

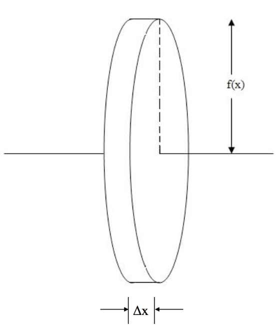
If we rotate an area under a function around the x-axis, it forms a 3-dimensional solid, called a volume of revolution.

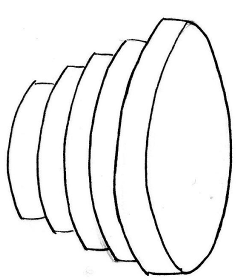




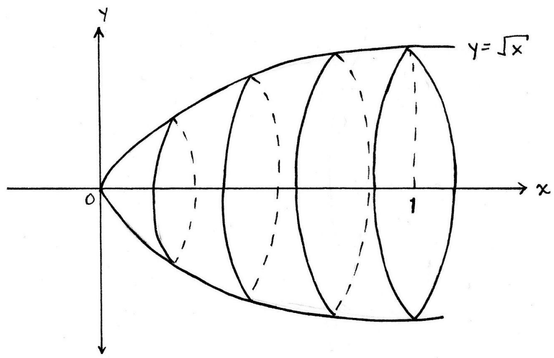
How can we find the volume of such an object?
Consider a small section. If it has width Δx (height) and height $f(x)$ (radius).
The volume of a cylinder is $V = \pi r^2 h = \pi f(x)^2 \Delta x$. If we have many of these, the volume is approximately

$$\begin{aligned}
V &\approx \sum_{i=1}^{n} \pi f(x_i)^2 \Delta x \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \pi f(x_i)^2 \Delta x \\
&= \int_a^b \pi f(x)^2 dx
\end{aligned}$$



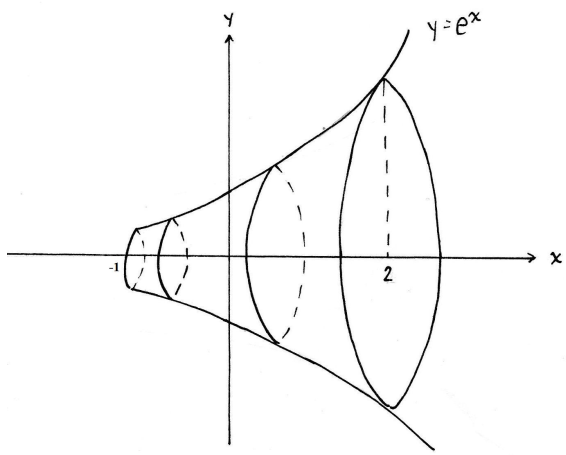


Example 23. Find the volume of the solid of revolution generated by rotating the region under the graph of $y = \sqrt{x}$ from x=0 to x=1 about the x-axis.



$$\begin{aligned}
V &= \int_0^1 \pi f(x)^2 dx \\
&= \int_0^1 \pi(\sqrt{x})^2 dx \\
&= \int_0^1 \pi x dx \\
&= \pi \frac{x^2}{2}\Big|_0^1 \\
&= \frac{\pi}{2} = 1.57 \ units^3
\end{aligned}$$

Example 24. Find the volume of the solid of revolution generated by rotating the region under $y = e^x$ from x=-1 to x=2 about the x-axis.

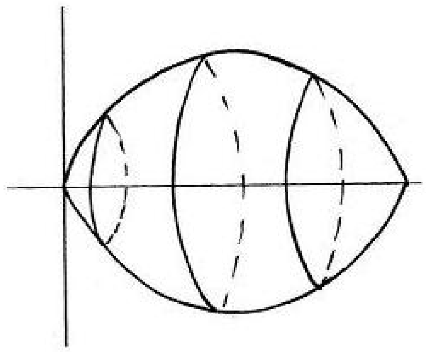


$$\begin{aligned}
V &= \int_{-1}^2 \pi(e^x)^2 dx \\
&= \int_{-1}^2 \pi e^{2x} dx \\
&= \frac{\pi e^{2x}}{2}\Big|_{-1}^2 \\
&= \frac{\pi}{2}(e^4 - e^{-2}) \\
&= 85.55 \ units^3
\end{aligned}$$

Example 25. Find the volume of the solid of revolution generated by rotating $y = \sin x$ from x=0 to x=π about the x-axis.

$Recall\ \cos(2x) = \cos^2(x) - \sin^2(x) = 1 - 2\sin^2(x)$

$and\ \sin(2x) = 2\sin(x)\cos(x)$



$$\begin{aligned} V &= \int_0^\pi \pi(\sin x)^2\ dx \\ &= \pi \int_0^\pi \sin^2(x)\ dx \\ &= \pi \int_0^\pi \frac{1-\cos(2x)}{2} dx \\ &= \frac{\pi}{2}\left[x - \frac{\sin 2x}{2}\right]_0^\pi \\ &= \frac{\pi}{2}\left[(\pi - 0) - (0 - 0)\right] \\ &= \frac{\pi^2}{2} = 4.93\ units^3 \end{aligned}$$

Exercise: Find the volume of the solid generated by rotating the positive part of $2x - x^2$ around the x-axis.
Answer: $\frac{16\pi}{15} = 3.35$

6 Improper Integrals

Proper integrals have the form $\int_a^b f(x)dx, \quad b < \infty$
Improper integrals have the form $\int_a^\infty f(x)dx$.
Why would we use this? Because infinity is a useful abstraction of "very big" or"very far."

Example 26. The rate of sales of a new product is $\frac{dS}{dt} = \frac{400}{(t+2)^2}$ where t is the time in weeks and S is amount in dollars. If the product were on sale forever, how much would it make?

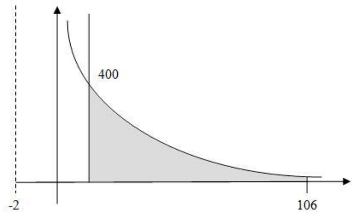
$$
\begin{aligned}
S_\infty &= \int_0^\infty \frac{400}{(t+2)^2} dt & u &= t+2 \\
&= \int_{t=0}^{t=\infty} 400u^{-2} du & \frac{du}{dt} &= 1 \\
&= \left[-400u^{-1}\right]_{t=0}^{\infty} \\
&= \left[-400(t+2)^{-1}\right]_0^{\infty} \\
&= \left[\frac{-400}{\infty+2}\right] - \left[\frac{-400}{0+2}\right] \Leftarrow WARNING! \\
(&= 0 + 200) \\
(&= \$200 \quad) & &\Leftarrow [Secret\ Knowledge]
\end{aligned}
$$

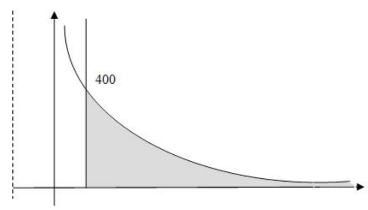
But we didn't really substitute ∞ into an equation. Instead, we take the limit so this is what we really do have:

$$
\begin{aligned}
S_\infty &= \lim_{T\to\infty} \int_0^T 400(t+2)^{-2} dt \\
&= \lim_{T\to\infty} \int_{t=0}^{t=T} 400u^{-2} du \\
&= \lim_{T\to\infty} \left[-400u^{-1}\right]_{t=0}^{t=T} \\
&= \lim_{T\to\infty} \left[-400(t+2)^{-1}\right]_0^T \\
&= \lim_{T\to\infty} \left[-\frac{400}{T+2} + \frac{400}{0+2}\right] \\
&= \lim_{T\to\infty} \left[-\frac{400}{T+2} + 200\right] \\
&= \$200
\end{aligned}
$$

But of course we never really wait forever. What if we waited a really long time, like two years (104 weeks)?

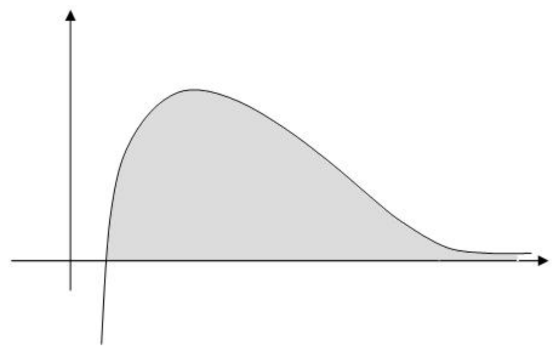
$$
\begin{aligned}
S(2) &= \int_0^{104} 400(t+2)^{-2}dt \\
&= \left[-400(t+2)^{-1}\right]_0^{104} \\
&= -\frac{400}{104+2} + 200 \\
&= -3.77 + 200 \\
&= \$196.23
\end{aligned}
$$





Example 27. $\int_1^\infty \frac{\ln t}{t} dt$
Try substitution:

$$\begin{aligned} u &= \ln t \\ \frac{du}{dt} &= \frac{1}{t} \\ dt &= t du \\ \int_{t=1}^{t=\infty} \frac{u}{t} t du &= \left. \frac{u^2}{2} \right|_{t=1}^{t=\infty} \\ &= \left. \frac{(\ln t)^2}{2} \right|_1^\infty \\ &= \lim_{T\to\infty} \left. \frac{(\ln t)^2}{2} \right|_1^T \\ &= \infty - 0 \end{aligned}$$



$$= \infty$$

Try integration by parts:

$$\begin{aligned} u &= \ln t & v' &= \frac{1}{t} \\ u' &= \frac{1}{t} & v &= \ln t \end{aligned}$$

$$\begin{aligned} I &= (\ln t)^2 \Big|_1^\infty - \int_1^\infty \frac{1}{t} \ln t dt \\ &= (\ln t)^2 \Big|_1^\infty - I \\ 2I &= (\ln t)^2 \Big|_1^\infty \\ I &= \left. \frac{(\ln t)^2}{2} \right|_1^\infty \\ &= \lim_{T\to\infty} \left. \frac{(\ln t)^2}{2} \right|_1^T \\ &= \lim_{T\to\infty} \left[\frac{(\ln T)^2}{2} - \frac{\ln(1)}{2} \right] \\ &= \infty - 0 \\ &= \infty \end{aligned}$$

If an integral is infinite, we say it diverges.
If an integral is finite, we say it converges.

Example 28. Does $\int_0^\infty e^{-2t}dt$ converge or diverge?

$$\begin{aligned}
\int_0^\infty e^{-2t}dt &= \lim_{T\to\infty}\int_0^T e^{-2t}dt \\
&= \lim_{T\to\infty} \left.\frac{e^{-2t}}{-2}\right|_0^T \\
&= \lim_{T\to\infty}\left[\frac{e^{-2T}}{-2} - \frac{e^0}{-2}\right] \\
&= \lim_{T\to\infty}\left[\frac{-1}{2e^{2T}} + \frac{1}{2}\right] \\
&= \frac{1}{2} \therefore \textit{ converges.}
\end{aligned}$$

Example 29. Does $\int_0^\infty \frac{x}{\sqrt{x+2}}dx$ converge or diverge?

$$\begin{aligned}
\textit{Let } u &= x+2 \\
du &= dx \\
\int_0^\infty \frac{x}{\sqrt{x+2}}dx &= \lim_{T\to\infty}\int_{x=0}^{x=T}\frac{x}{\sqrt{u}}du \\
&= \lim_{T\to\infty}\int_{x=0}^{x=T}\frac{u-2}{\sqrt{u}}du \qquad (\textit{since } u = x+2, x = u-2) \\
&= \lim_{T\to\infty}\int_{x=0}^{x=T}\left(\frac{u}{\sqrt{u}} - \frac{2}{\sqrt{u}}\right)du \\
&= \lim_{T\to\infty}\int_{x=0}^{x=T}\frac{u}{u^{\frac{1}{2}}} - \frac{2}{u^{\frac{1}{2}}}du \\
&= \lim_{T\to\infty}\int_{x=0}^{x=T} u^{\frac{1}{2}} - 2u^{-\frac{1}{2}}du \\
&= \lim_{T\to\infty}\left[\frac{2u^{\frac{3}{2}}}{3} - 4u^{\frac{1}{2}}\right]_{x=0}^{x=T} \\
&= \lim_{T\to\infty}\left[\frac{2(x+2)^{\frac{3}{2}}}{3} - 4(x+2)^{\frac{1}{2}}\right]_{x=0}^{x=T} \\
&= \lim_{T\to\infty}\left[\frac{2(T+2)^{\frac{3}{2}}}{3} - 4(T+2)^{\frac{1}{2}}\right] - \left[\frac{2(2)^{\frac{3}{2}}}{3} - 4(2)^{\frac{1}{2}}\right] \\
&= \lim_{T\to\infty}(T+2)^{\frac{1}{2}}\left[\frac{2}{3}(T+2) - 4\right] - \left[\frac{2(2)^{\frac{3}{2}}}{3} - 4(2)^{\frac{1}{2}}\right] \\
&= \infty(\infty - 4) - \left[\frac{2(2)^{\frac{3}{2}}}{3} - 4(2)^{\frac{1}{2}}\right] \\
&= \infty \qquad \therefore \textit{ diverges}
\end{aligned}$$

Example 30. $\int_0^\infty \frac{1}{2e^{2x}}dx$

$$\begin{aligned}
\int_0^\infty \frac{1}{2e^{2x}}dx &= \frac{1}{2}\lim_{T\to\infty}\int_0^T e^{-2x}dx \\
&= \frac{1}{2}\lim_{T\to\infty} -\frac{1}{2}e^{-2x}\Big|_0^T \\
&= -\frac{1}{4}\lim_{T\to\infty}\left[e^{-2T}-1\right] \\
&= -\frac{1}{4}(0-1) \\
&= \frac{1}{4}
\end{aligned}$$

Example 31. The rate at which you learn is $\frac{dC}{dt} = 50t^2e^{-t}$. How much will you learn if you study forever?

$C_\infty = \int_0^\infty 50t^2e^{-t}dt$

Try taking the constant out - this does not simplify.

Try substitution - nothing cancels.

Try integration by parts:

$$\begin{aligned}
&u = 50t^2 \qquad && v' = e^{-t} \\
&u' = 100t \qquad && v = -e^{-t}
\end{aligned}$$

$$C_\infty = -50t^2e^{-t} + 100\int_0^\infty te^{-t}dt$$

Use integration by parts again

$$\begin{aligned}
&u = t \qquad && v' = e^{-t} \\
&u' = 1 \qquad && v = -e^{-t}
\end{aligned}$$

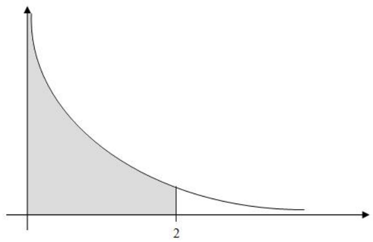
$$\begin{aligned}
C_\infty &= -50t^2e^{-t} + 100\left[-te^{-t} + \int_0^\infty e^{-t}dt\right] \\
&= \left[-50t^2e^{-t} - 100te^{-t} - 100e^{-t}\right]_0^\infty \\
&= \lim_{T\to\infty}\left[(-50T^2e^{-T} - 100Te^{-T} - 100e^{-T}) - (0-0-100)\right] \\
&= \lim_{T\to\infty}\left[-\frac{50T^2}{e^T} - \frac{100T}{e^T} - \frac{100}{e^T} + 100\right] \\
&= \frac{\infty}{\infty} - \frac{\infty}{\infty} - 0 + 100 \therefore \text{ use l'Hôpital's rule on the first two terms} \\
&= \lim_{T\to\infty}\left[-\frac{100T}{e^T} - \frac{100}{e^T} - \frac{100}{e^T}\right] + 100 \\
&= \frac{\infty}{\infty} - 0 - 0 + 100 \therefore \text{use l'Hôpital's rule again} \\
&= \lim_{T\to\infty}\left[-\frac{100}{e^T} - \frac{100}{e^T} - \frac{100}{e^T}\right] + 100 \\
&= -0 - 0 - 0 + 100 \\
&= 100\%
\end{aligned}$$

Therefore if you study forever you will learn 100 percent.

7 Infinite Integrands

We want to find $\int_0^2 \frac{1}{\sqrt{x}}dx$. Why is this a problem? Because $\frac{1}{\sqrt{x}}$ is not defined at 0.
But let's try to do what we did with improper integrals:

$$\begin{aligned}
\int_0^2 \frac{1}{\sqrt{x}} dx &= \lim_{\epsilon \to 0^+} \int_\epsilon^2 \frac{1}{\sqrt{x}} dx \\
&= \lim_{\epsilon \to 0^+} \int_\epsilon^2 x^{-\frac{1}{2}} dx \\
&= \lim_{\epsilon \to 0^+} \left[2x^{\frac{1}{2}}\right]_\epsilon^2 \\
&= \lim_{\epsilon \to 0^+} (2\sqrt{2} - 2\sqrt{\epsilon}) \\
&= 2\sqrt{2} - 0 \\
&= 2\sqrt{2} = 2.828 \text{ where } 0^+ \text{means the limit from the right.}
\end{aligned}$$



Therefore, even though the function goes to ∞ at 0, the area is well-defined. (It's like the improper integral, only turned sideways.)

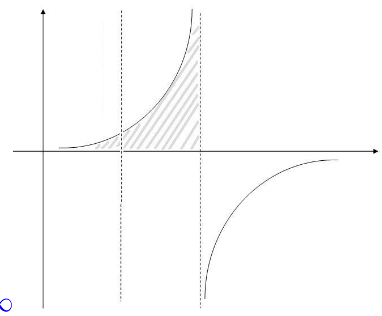
Example 32. $\int_0^3 (-4x^{-2} - 3x^{-1} + 1) dx$

$$\begin{aligned}
\int_0^3 (-4x^{-2} - 3x^{-1} + 1) dx &= \lim_{\epsilon \to 0^+} \left[4x^{-1} - 3\ln x + x\right]_\epsilon^3 \\
&= \lim_{\epsilon \to 0^+} \left[\frac{4}{3} - 3\ln 3 + 3\right] - \left[\frac{4}{\epsilon} - 3\ln \epsilon + \epsilon\right] \\
&= \lim_{\epsilon \to 0^+} \left[\frac{13}{3} - 3\ln 3 - \frac{4}{\epsilon} + 3\ln \epsilon - \epsilon\right] \\
&= \frac{13}{3} - 3\ln 3 - \infty + 3(-\infty) - 0 \\
&= -\infty
\end{aligned}$$

Example 33. $\int_0^1 \frac{1}{1-x} dx$

$$\int_0^1 \frac{1}{1-x}dx = \lim_{\epsilon \to 1^-} \int_0^{\epsilon} \frac{1}{1-x}dx$$

$$\begin{aligned}
&\textit{Substitute } u = 1 - x \\
&\frac{du}{dx} = -1 \\
&= \lim_{\epsilon \to 1^-} \int_{x=0}^{x=\epsilon} -\frac{1}{u}du \\
&= \lim_{\epsilon \to 1^-} (-\ln u) \Big|_{x=0}^{x=\epsilon} \\
&= \lim_{\epsilon \to 1^-} -\ln(1-x) \Big|_0^{\epsilon} \\
&= \lim_{\epsilon \to 1^-} (-\ln(1-\epsilon) + \ln(1))
\end{aligned}$$



$$= -\ln 0 + 0 = +\infty$$

Example 34. $\int_{-1}^{3} -\frac{1}{x^2}dx$

The function $-\frac{1}{x^2}$ has domain $\{x \in \mathbb{R} : x \neq 0\}$, which means we need to split the integral at 0.

$$\begin{aligned}
\int_{-1}^{3} -\frac{1}{x^2}dx &= \lim_{\epsilon \to 0^-} \int_{-1}^{\epsilon} -\frac{1}{x^2}dx + \lim_{\epsilon \to 0^+} \int_{\epsilon}^{3} -\frac{1}{x^2}dx \\
&= \lim_{\epsilon \to 0^-} \frac{1}{x}\Big|_{-1}^{\epsilon} + \lim_{\epsilon \to 0^+} \frac{1}{x}\Big|_{\epsilon}^{3} \\
&= \lim_{\epsilon \to 0^-} \left(\frac{1}{\epsilon} + 1\right) + \lim_{\epsilon \to 0^+} \left(\frac{1}{3} - \frac{1}{\epsilon}\right) \\
&= (-\infty + 1) + \left(\frac{1}{3} - \infty\right) \\
&= -\infty
\end{aligned}$$

8 Partial Fractions

First introductory example

We don't know how to integrate the fraction $\int \frac{x}{x+2}dx$, but we can write the fraction in a simpler way and use known rules to find the integral as follows:

$$\int \frac{x}{x+2}dx = \int \frac{(x+2)-2}{x+2}dx = \int \left[1 - \frac{2}{x+2}\right] dx = x - 2\ln|x+2| + C.$$

Second introductory example

We don't know integrate the fraction $\int \frac{3x-2}{x(x-2)}dx$. But if we simplify the fraction as

$$\frac{3x-2}{x(x-2)} = \frac{1}{x} + \frac{2}{x-2}$$

(check this!) then we can integrate as follows:

$$\int \frac{3x-2}{x(x-2)}dx = \int \left(\frac{1}{x} + \frac{2}{x-2}\right) dx = \ln|x| + 2\ln|x-2| + C.$$

General Idea

Rational functions are fractions of polynomials, i.e., if $P(x)$ and $Q(x)$ are polynomials, then $P(x)/Q(x)$ is called a rational function. We already know how to integrate some of them, namely the following building blocks (you need to know these!)

$$\begin{aligned} \int \frac{1}{x+a}dx &= \ln|x+a| + C, \\ \int \frac{1}{x^2+1}dx &= \arctan(x) + C = \tan^{-1}(x) + C, \\ \int \frac{x}{x^2+1}dx &= \frac{1}{2}\ln(x^2+1) + C. \end{aligned}$$

(You don't have to memorize the last one; you could use substitution to solve it.)

If we can split a rational function into sums of these building blocks, then we can integrate easily. The goal of this section is to find a technique to integrate (find antiderivatives of) all rational functions. We only consider cases where $\deg(Q) \leq 2$, i.e., the highest power of x in the denominator is no more than 2. The idea is to decompose a rational function into a sum of simpler rational functions, namely the three examples above, which we know how to integrate.

Recipe for partial fractions

To find the integral of a rational function $P(x)/Q(x)$, follow these steps.

1. If $\deg(P) \geq \deg(Q)$ then use long division to split the rational function into several parts. Now assume that $\deg(P) < \deg(Q)$.

2. If $Q(x) = ax^2 + bx + c = a(x-x_1)(x-x_2)$ has two distinct real roots, the one can find numbers A, B such that
$$\frac{P(x)}{Q(x)} = \frac{1}{a}\left[\frac{A}{x-x_1} + \frac{B}{x-x_2}\right].$$
Then use the natural logarithm to integrate the two terms.

3. If $Q(x) = ax^2 + bx + c = a(x-x_1)^2$ has only one real root, the one can find numbers A, B such that
$$\frac{P(x)}{Q(x)} = \frac{1}{a}\left[\frac{A}{x-x_1} + \frac{B}{(x-x_1)^2}\right].$$
Then one can integrate using substitution, the logarithm, and direct integration.

4. If $Q(x) = ax^2 + bx + c$ has no real roots, then complete the square to get
$$Q(x) = a\left[\left(x - \frac{b}{2a}\right)^2 + \frac{c}{a} - \left(\frac{b}{2a}\right)^2\right] = a[(x-A)^2 + B].$$
Then use the natural logarithm and the arctan to integrate the two terms (potentially substitute first).

We illustrate each of these cases with examples.

Example 35. $P(x) = x^2 + 1, Q(x) = x - 1.$

We have $\deg(P) = 2 > 1 = \deg(Q)$, so we need to do long division. We find

$$x^2 + 1 = (x - 1)(x + 1) + 2.$$

Therefore

$$\int \frac{x^2+1}{x-1} dx = \int \left[x + 1 + \frac{2}{x-1}\right] dx = \frac{x^2}{2} + x + 2\ln|x-1| + C.$$

Example 36. $P(x) = 2x^3 + 3x^2 + 2x + 4, Q(x) = x^2 + 1.$

Again, since $\deg(P) = 3 > 2 = \deg(Q)$, we need to do long division. We find

$$2x^3 + 3x^2 + 2x + 4 = (x^2 + 1)(2x + 3) + 1.$$

Therefore

$$\int \frac{2x^3 + 3x^2 + 2x + 4}{x^2+1} dx = \int \left[2x + 3 + \frac{1}{x^2+1}\right] dx = x^2 + 3x + \arctan(x) + C.$$

Example 37. $P(x) = 2x + 1, Q(x) = x^2 + x - 2.$

This time, $\deg(P) = 1 < 2 = \deg(Q)$, so no long division is necessary. Instead, we factor Q as $Q(x) = (x - 1)(x + 2)$, so that

$$\frac{2x+1}{x^2+x-2} = \frac{2x+1}{(x-1)(x+2)}.$$

On the other hand, for two numbers, A, B, we find

$$\frac{A}{x-1} + \frac{B}{x+2} = \frac{(A+B)x + 2A - B}{(x-1)(x+2)}.$$

Comparing with the expression above, we find that $A + B = 2$ and $2A - B = 1$. Hence, $A = B = 1$. Then we integrate

$$\int \left[\frac{2x+1}{x^2+x-2}\right] dx = \int \left[\frac{1}{x-1} + \frac{1}{x+2}\right] dx = \ln|x-1| + \ln|x+2| + C.$$

Example 38. $P(x) = x + 5, Q(x) = x^2 - 4x + 4.$

Again, $\deg(P) = 1 < 2 = \deg(Q)$, so no long division is necessary. But $Q(x) = (x-2)^2$, has only a single root, i.e.,

$$\frac{x+5}{x^2-4x+4} = \frac{x+5}{(x-2)^2}.$$

On the other hand, for two numbers, A, B, we find

$$\frac{A}{x-2} + \frac{B}{(x-2)^2} = \frac{Ax - 2A + B}{(x-2)^2}.$$

Comparing with the expression above, we find that $A = 1$ and $-2A + B = 5$. Hence, $A = 1, B = 7$. Then we integrate

$$\int \left[\frac{x+5}{x^2-4x+4}\right] dx = \int \left[\frac{1}{x-2} + \frac{7}{(x-2)^2}\right] dx = \ln|x-2| - \frac{7}{x-2} + C.$$

Example 39. $P(x) = 3x + 2, Q(x) = x^2 - 2x + 5.$

No long division necessary. However, Q has no real roots. We complete the square

$$Q(x) = x^2 - 2x + 5 = x^2 - 2x + 1 - 1 + 5 = (x-1)^2 + 4.$$

Now we write

$$\int \frac{3x+2}{x^2-2x+5}dx = \int \frac{3x+2}{(x-1)^2+4}dx = \frac{1}{4}\int \frac{3x+2}{(\frac{x-1}{2})^2+1}dx.$$

This is a case for substitution. We choose $u = \frac{x-1}{2}$ so that $x = 2u+1$ and $dx = 2du$. Then we get

$$\frac{1}{4}\int \frac{3x+2}{(\frac{x-1}{2})^2+1}dx = \frac{1}{2}\int \frac{6u}{u^2+1}du + \frac{1}{2}\int \frac{5}{u^2+1}du.$$

The first of these integrals requires another substitution, $w = u^2+1$, the second is again an arctan. With this we find

$$\frac{1}{2}\int \frac{6u}{u^2+1}du + \frac{1}{2}\int \frac{5}{u^2+1}du = \frac{1}{2}\int \frac{3}{w}dw + \frac{1}{2}\int \frac{5}{u^2+1}du = \frac{3}{2}\ln|w| + \frac{5}{2}\arctan(u) + C.$$

After back-substituting, we find that the integral with respect to x is given by

$$\frac{3}{2}\ln\left|\frac{x^2}{4} - \frac{x}{2} + \frac{5}{4}\right| + \frac{5}{2}\arctan\left(\frac{x-1}{2}\right) + C.$$

Example 40. $P(x) = x^2 - 2, Q(x) = x^2 - 3x + 2.$

Long division first, or the simpler way

$$\frac{x^2-2}{x^2-3x+2} = \frac{x^2-3x+2+3x-4}{x^2-3x+2} = 1 + \frac{3x-4}{x^2-3x+2}.$$

Now, the denominator is $Q(x) = (x-1)(x-2)$, hence we set the partial fractions as

$$\frac{A}{x-1} + \frac{B}{x-2} = \frac{(A+B)x - (2A+B)}{x^2-3x+2}.$$

Hence, we need $A + B = 3$ and $2A + B = 4$, which is given by $A = 1, B = 2$. Now we can integrate

$$\int \frac{x^2-2}{x^2-3x+2}dx = \int \left(1 + \frac{1}{x-1} + \frac{2}{x-2}\right) dx = x + \ln|x-1| + 2\ln|x-2| + C.$$

9 Differential Equations

Your town is suffering an epidemic. Your chances of catching the disease are proportional to the probability you meet a carrier of the disease. That is: (probability that you are in a place at a given time) $\times$ (probability that a carrier is also in given place at given time).

Outline of the process:

- Word problem
- Translate into equations
- Sharpen up equations
- Initial conditions
- One-variable problem

- Separation of variables
- Solve
- Find constant of integration
- Sketch solution
- Biological interpretation

S - susceptible individuals
I - infected individuals

$$\frac{dS}{dt} \propto -SI$$
$$\frac{dI}{dt} \propto SI$$

I increases due to encounters (more people get sick).
S decreases by the same amount (susceptible people become infected).

Sharper: replace "$\propto$" with "$= \beta$". Let's suppose that 10% of people are infected initially.

$$\begin{aligned}
&S' = -\beta SI \\
&I' = \beta SI \\
&S' + I' = 0 \\
&S + I = N \ (constant) \\
&\therefore S = N - I \\
&I' = \beta(N-I)I \qquad \text{which is a single autonomous differential equation.}
\end{aligned}$$

How do we solve this? Using separation of variables.

Steps:
1. Put state variable on one side and time variable (including dt) on the other.
2. Integrate both sides.
3. Set integrals equal to each other.
4. Combine two integrating constants into one.
5. Solve for the state variable (may rewrite the constant in a more convenient form.)
6. Solve for the constant using the intial condition.

$$
\begin{aligned}
\frac{dI}{dt} &= \beta(N-I)I \\
\frac{dI}{(N-I)I} &= \beta dt \\
\int \frac{dI}{(N-I)I} &= \beta \int dt \\
\frac{1}{(N-I)I} &= \frac{A}{N-I} + \frac{B}{I} \\
1 &= AI + B(N-I)
\end{aligned}
$$

$$I = 0 \qquad 1 = BN \qquad B = \frac{1}{N}$$

$$I = N \qquad 1 = AN \qquad A = \frac{1}{N}$$

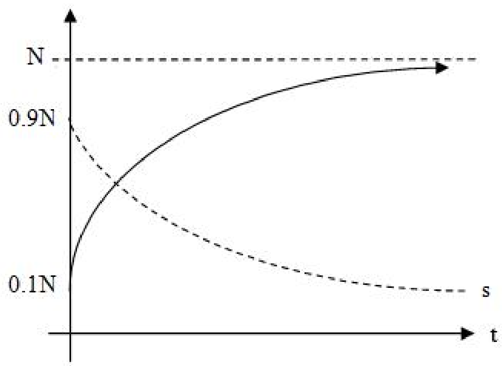
$$
\begin{aligned}
\int \left(\frac{1}{N}\frac{1}{N-I} + \frac{1}{N}\frac{1}{I}\right) dI &= \beta \int dt \\
\frac{1}{N}\left(-\ln|N-I|\right) + \frac{1}{N}\ln|I| &= \beta t + c \leftarrow \ \textit{only one constant.} \\
\frac{1}{N}\ln\frac{I}{N-I} &= \beta t + c \\
\ln\frac{I}{N-I} &= \beta N t + cN \\
\frac{I}{N-I} &= e^{\beta N t}e^{cN} = ke^{\beta N t}
\end{aligned}
$$

$$
\begin{aligned}
&I = (N-I)ke^{\beta N t} \\
&I = Nke^{\beta N t} - Ike^{\beta N t} \\
&I(1+ke^{\beta N t}) = Nke^{\beta N t} \\
&I = \frac{Nke^{\beta N t}}{1+ke^{\beta N t}}
\end{aligned}
$$

$$I(0) = \frac{Nk}{1+k} = \frac{N}{10} \qquad \textit{(10\% of population infected)}$$

$$
\begin{aligned}
10k &= 1+k \\
9k &= 1 \\
k &= \frac{1}{9}
\end{aligned}
$$

$$
\begin{aligned}
I &= \frac{\frac{N}{9}e^{\beta N t}}{1+\frac{e^{\beta N t}}{9}} = \frac{Ne^{\beta N t}}{9+e^{\beta N t}} \\
\lim_{t\to\infty} I(t) &= \lim_{t\to\infty}\frac{Ne^{\beta N t}}{9+e^{\beta N t}} \\
&= \lim_{t\to\infty}\frac{\beta N^2 e^{\beta N t}}{\beta N e^{\beta N t}} \quad \text{using l'Hopital's rule} \\
&= N
\end{aligned}
$$



Therefore, eventually everyone gets infected.

Exercise: Check $I(0) = 0.1N$ and I satisfies $I' = \beta(N - I)I$.

Example 41. Suppose $\frac{dx}{dt} = x + x^2$.

a) Set $y = \frac{1}{x}$ and find a differential equation for y.

b) Solve for x if x(0)=1.

Solution: a)

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}\\ \frac{dy}{dt} &= \frac{dy}{dx}\frac{dx}{dt}\\ &= -\frac{1}{x^2}(x + x^2)\\ &= -\frac{1}{x} - 1\\ &= -y - 1\end{aligned}$$

Therefore we have transformed a nonlinear equation into a linear equation.

b)

$$\begin{aligned}\frac{dy}{y+1} &= -dt\\ \int \frac{1}{y+1}dy &= -\int dt\\ \ln|y+1| &= -t + c\\ y + 1 &= e^{-t+c} = Ae^{-t}\\ y &= Ae^{-t} - 1\\ \frac{1}{x} &= Ae^{-t} - 1\\ x &= \frac{1}{Ae^{-t} - 1}\\ x(0) &= \frac{1}{A-1} = 1\\ A - 1 &= 1 \to A = 2\\ \therefore x &= \frac{1}{2e^{-t} - 1}\end{aligned}$$

What if we solve directly?

$$\begin{aligned}
\frac{dx}{x+x^2} &= dt \\
\int \frac{dx}{x+x^2} &= \int dt \\
\frac{1}{x(1+x)} &= \frac{A}{x} + \frac{B}{1+x} \\
1 &= A(1+x) + Bx \\
x = 0 & \qquad\qquad 1 = A \\
x = -1 & \qquad\qquad 1 = -B \\
\int \left(\frac{1}{x} - \frac{1}{1+x} \right) dx &= \int dt \\
\ln|x| - \ln|1+x| &= t + c \\
\ln\left|\frac{x}{1+x}\right| &= t + c \\
\frac{x}{1+x} &= e^{t+c} = ke^t \\
x &= ke^t(1+x) \\
x(1-ke^t) &= ke^t \\
x &= \frac{ke^t}{1-ke^t} \\
x(0) &= \frac{k}{1-k} = 1 \to k = 1-k \to 2k = 1 \to k = \frac{1}{2} \\
x &= \frac{\frac{1}{2}e^t}{1-\frac{1}{2}e^t} = \frac{e^t}{2-e^t}
\end{aligned}$$

Is this the same answer? Yes since $x = \frac{e^t}{2-e^t}\frac{e^{-t}}{e^{-t}} = \frac{1}{2e^{-t}-1}$.

Example 42. $\frac{dx}{dt} = \frac{x}{2x-1} \qquad x(0) = 1$

$$\begin{aligned}
\frac{2x-1}{x} dx &= dt \\
\int \frac{2x-1}{x} dx &= \int dt \\
\int 2 - \frac{1}{x} dx &= t + c \\
2x - \ln|x| &= t + c
\end{aligned}$$

In this case we can't find the solution explicitly, because we can't isolate x. But we can still find c.

$$x(0) = 1 \Rightarrow 2(1) - \ln(1) = 0 + c \Rightarrow 2 = c$$
$$\therefore 2x - \ln|x| = t + 2 \text{ is the implicit solution.}$$

Exercise: Solve $x' = \frac{x^3-3x}{t}$ with $x(1) = 2$

10 Equilibria

What is a derivative? It is a rate of change. The system is at equilibrium if there is no change. That is, the derivative is zero.

Example 43. $\frac{dx}{dt} = x + x^2$
We know that $x = \frac{1}{Ae^{-t}-1} = \frac{e^t}{A-e^t}$

$$\lim_{t\to\infty} x = -1 \qquad\qquad \lim_{t\to-\infty} x = 0 \; with \; A \neq 0$$

Finding the equilibria:

$$\begin{aligned} &x + x^2 = 0 \\ &x(1+x) = 0 \\ &x = 0, \quad x = -1 \\ &A = 0 \Rightarrow x = -1 \; always \therefore \; equilibrium. \\ &A = \infty \Rightarrow x = 0 \; always \therefore \; equilibrium. \end{aligned}$$

Example 44. Disease example: $I' = \beta(N - I)I$

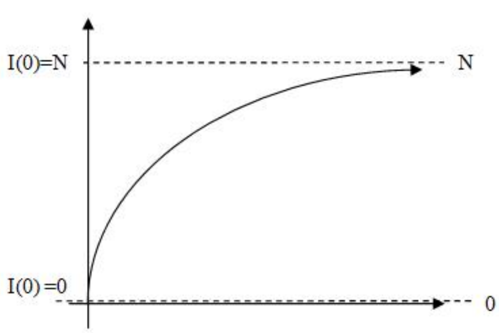
$$\begin{aligned} &I' = 0 \Rightarrow \beta(N-I)I = 0 \\ &\qquad \beta = 0 \qquad N - I = 0 \quad I = 0 \\ &\qquad \beta = 0 \qquad I = N \qquad I = 0 \\ &\quad \nearrow \qquad\qquad \uparrow \qquad\qquad \nwarrow \end{aligned}$$

no transmission   everyone infected   nobody infected

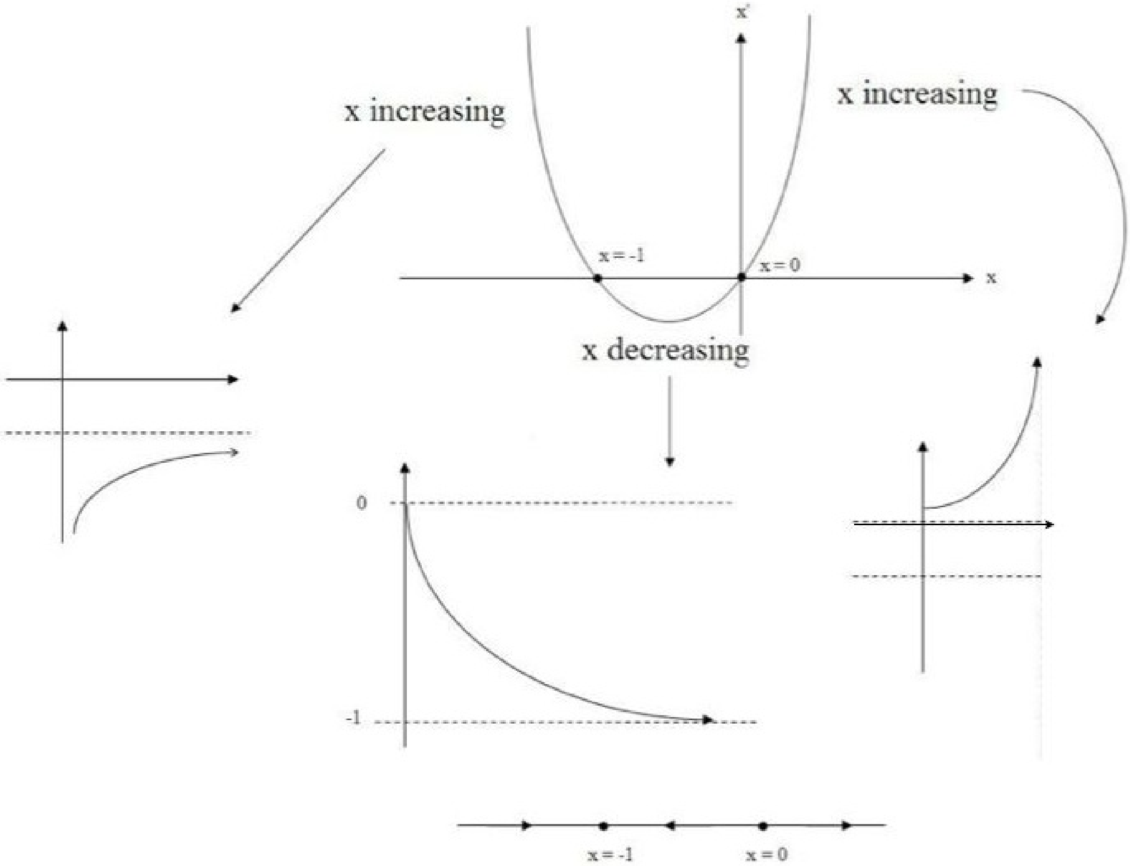
$$Solution: I = \frac{Nke^{\beta Nt}}{1+ke^{\beta Nt}}$$

$$\beta = 0 \Rightarrow I = \frac{Nk}{1+k} \qquad \text{Recall that } \lim_{t\to\infty} I = N$$

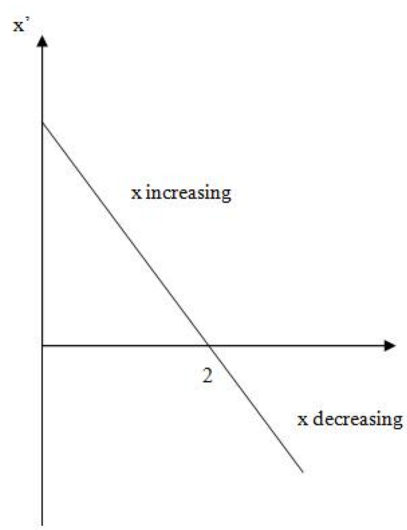
Thus there is no change over time.



Example 45. $x' = x + x^2$



Example 46. $\frac{dx}{dt} = 2 - x$



$Equilibrium: \; 2 - x = 0 \;\Rightarrow\; x = 2.$

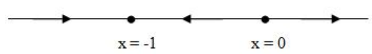
11 Stability

Definition 11.1. *An equilibrum is* <u>*stable*</u> *if solutions that begin near the equilibrium approach the equilibrium. An equilibrium is* <u>*unstable*</u> *if solutions that begin near the equilibrium move away from the equilibrium.*

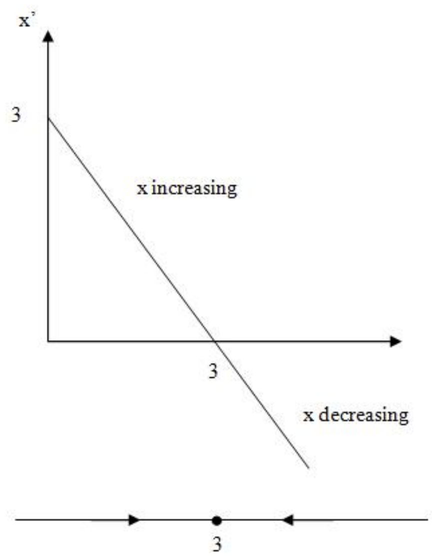
$\therefore$ x=2 is a stable equilibrium in the previous example.

Example 47. $\frac{dx}{dt} = x + x^2$

$$x = -1 \; is \; stable$$
$$x = 0 \; is \; unstable.$$



Example 48. $\frac{dx}{dt} = 3 - x$
Equilibrium: $3 - x = 0 \Rightarrow x = 3$
It looks like $x = 3$ is stable. How can we know for sure?

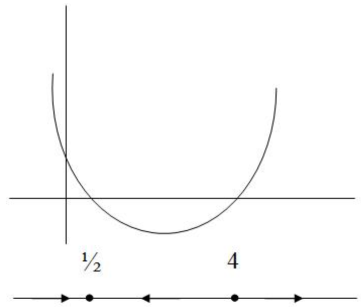


$$\begin{aligned}
\frac{dx}{3-x} &= dt \\
\int \frac{dx}{3-x} &= \int dt \\
-\ln|3-x| &= t + c \\
\ln|3-x| &= -t - c \\
3 - x &= e^{-t-c} \\
x &= 3 - Ae^{-t} \\
\lim_{t\to\infty} x &= 3 - A \lim_{t\to\infty} e^{-t} \\
&= 3 - 0 \\
&= 3 \;\; \text{regardless of what A is.}
\end{aligned}$$

Example 49. $\frac{dx}{dt} = 2x^2 - 9x + 4$. Determine equilibria and their stability.

Equilibria :

$$2x^2 - 9x + 4 = 0$$
$$(2x-1)(x-4) = 0$$



$$x = \frac{1}{2},\ x = 4$$

$x = \frac{1}{2}$ is stable and $x = 4$ is unstable.

What if we can't sketch the graph?

Stability Theorem Suppose y^* is an equilibrium of $\frac{dy}{dt} = f(y)$. Then

y^* is stable if $f'(y^*) < 0$
y^* is unstable if $f'(y^*) > 0$

Example 50. $\frac{dx}{dt} = 2x^2 - 9x + 4$. Determine all equilibria and their stability.

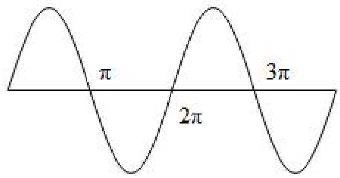
We know that $x^* = \frac{1}{2}, 4$.

$$f(x) = 2x^2 - 9x + 4$$
$$f'(x) = 4x - 9$$
$$f'\left(\frac{1}{2}\right) = 2 - 9 = -7 < 0 \qquad \therefore \frac{1}{2} \textit{ is stable.}$$
$$f'(4) = 16 - 9 = 7 > 0 \qquad \therefore 4 \textit{ is unstable.}$$

Example 51. $\frac{dx}{dt} = \sin x, \quad x \geqslant 0$. Determine all equilibria and their stability.

Equilibia: $\sin x = 0 \rightarrow \ x = 0, \pi, 2\pi, 3\pi, ...$

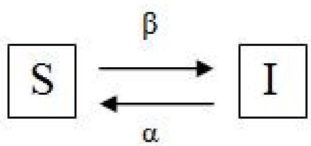
$$
\begin{aligned}
f'(x) &= \cos x \\
f'(0) &= \cos 0 = 1 \\
f'(\pi) &= \cos \pi = -1 \\
f'(2\pi) &= \cos 2\pi = 1 \\
f'(3\pi) &= \cos 3\pi = -1 \\
&\vdots
\end{aligned}
$$



$\therefore$ the even multiples of π are unstable equilibria and the odd multiples of π are stable equilibria.

Example 52. SIS epidemic. In this disease people get sick as before, but they also recover after a while.

$$
\begin{aligned}
S' &= \alpha I - \beta SI \\
I' &= \beta SI - \alpha I \\
S' + I' &= 0 \to S + I = N \\
I' &= \beta(N - I)I - \alpha I
\end{aligned}
$$



In this case we can't solve as we did before, but we can still discover pertinent information.

Equilibria:

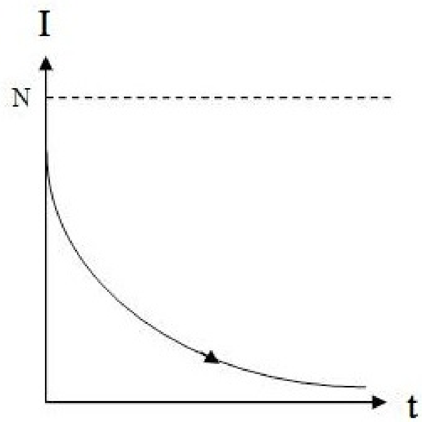
$$
\begin{aligned}
&[\beta(N - I) - \alpha]I = 0 \\
&I = 0 \qquad \beta(N - I) - \alpha = 0 \\
&\qquad\qquad N - I = \frac{\alpha}{\beta} \\
&\qquad\qquad I = N - \frac{\alpha}{\beta} = \frac{\beta N - \alpha}{\beta}
\end{aligned}
$$

Two cases:

$$
\begin{aligned}
&(i)\beta N - \alpha < 0 \\
&(ii)\beta N - \alpha > 0 \\
&f(I) = \beta(N - I)I - \alpha I \\
&\qquad = \beta NI - \beta I^2 - \alpha I \\
&f'(I) = \beta N - 2\beta I - \alpha
\end{aligned}
$$

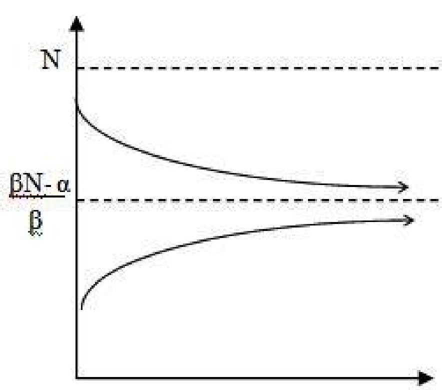
Case (i) I=0 is the only realistic equilibrium.

$$f'(0) = \beta N - \alpha < 0 \text{ in this case only!}$$
$$\therefore \; I = 0 \text{ is stable. That is, the disease dies out on its own.}$$



Case(ii) Two equilibria: I=0 and $I = \frac{\beta N - \alpha}{\beta}$

$$f'(0) = \beta N - \alpha > 0 \therefore \; I = 0 \textit{ is unstable in this case.}$$
$$\begin{aligned} f'\left(\frac{\beta N - \alpha}{\beta}\right) &= \beta N - 2\beta\left[\frac{\beta N - \alpha}{\beta}\right] - \alpha \\ &= \beta N - 2(\beta N - \alpha) - \alpha \\ &= \beta N - 2\beta N + 2\alpha - \alpha \\ &= -\beta N + \alpha \\ &= -(\beta N - \alpha) < 0 \end{aligned}$$
$$\therefore \frac{\beta N - \alpha}{\beta} \text{ is stable in this case.}$$

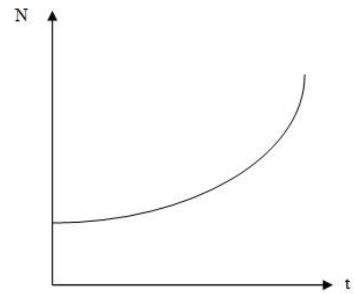


Therefore, if $\beta N - \alpha < 0$, ie recovery rate is high or transmission probability is low, then the disease dies out.

If $\beta N - \alpha > 0$, ie recovery rate is low or transmission probability is high, then the disease becomes established in the population. It doesn't infect everyone, but a certain proportion of people are always infected at any given time. Such a disease is called <u>endemic.</u>

12 The Logistic Equation

Many populations grow quickly at first but slow their growth when they run out of space. For example, rabbits on an island.
Unlimited growth can be written as $\frac{dN}{dt} = rN \Rightarrow N = Ae^{rt}$.



Slight limitation on growth $\frac{dN}{dt} = rN - bN^2$

$$= rN(1 - \tfrac{b}{r}N)$$
$$= rN(1 - \tfrac{N}{K}) \qquad K = \tfrac{r}{b} \text{ (we'll see why later)}$$

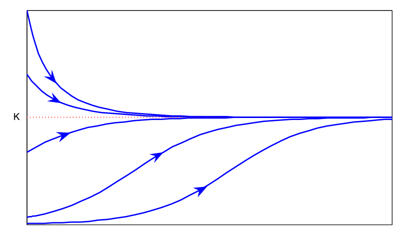
Separation of variables:

$$\frac{dN}{N(1-\frac{N}{K})} = rdt$$
$$\frac{1}{N(1-\frac{N}{K})} = \frac{A}{N} + \frac{B}{1-\frac{N}{K}}$$
$$1 = A(1 - \frac{N}{K}) + BN$$
$$N = 0 \qquad 1 = A$$
$$N = K \qquad 1 = BK \qquad B = \frac{1}{K}$$

$$\int \left(\frac{1}{N} + \frac{1}{K(1-\frac{N}{K})} \right) dN = \int rdt$$
$$\int \left(\frac{1}{N} + \frac{1}{K-N} \right) dN = \int rdt$$
$$\ln N - \ln{(K-N)} = rt + c$$
$$\ln \frac{N}{K-N} = rt + c$$
$$\frac{N}{K-N} = De^{rt}$$
$$N = (K-N)De^{rt}$$
$$N(1 + De^{rt}) = KDe^{rt}$$
$$N = \frac{KDe^{rt}}{1+De^{rt}}$$

$$
\begin{aligned}
N(0) = N_0 \Rightarrow N_0 &= \frac{KD}{1+D} \\
N_0(1+D) &= KD \\
N_0 + N_0 D &= KD \\
N_0 &= (K - N_0)D \\
D &= \frac{N_0}{K - N_0}
\end{aligned}
$$

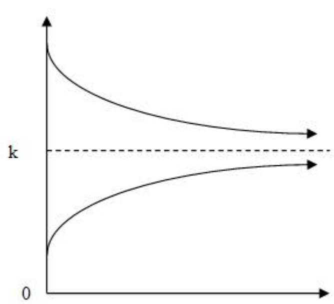
$$
\begin{aligned}
N &= \frac{\frac{KN_0e^{rt}}{K-N_0}}{1+\frac{N_0e^{rt}}{K-N_0}} \\
&= \frac{KN_0e^{rt}}{K - N_0 + N_0e^{rt}} \\
&= \frac{Ke^{rt}}{\frac{K}{N_0} - 1 + e^{rt}} \\
&= \frac{K}{\left(\frac{K}{N_0} - 1\right)e^{-rt} + 1}
\end{aligned}
$$



$\lim_{t\to\infty} N(t) = K$ K is called the carrying capacity.

Or we have the equilibria:

$$
\begin{aligned}
&N = 0, 1 - \frac{N}{K} = 0 \Rightarrow N = K \\
&f'(N) = r\left(1 - \frac{N}{K}\right) - \frac{rN}{K} \\
&f'(0) = r > 0 \therefore\ \textit{unstable} \\
&f'(K) = -r < 0 \therefore\ \textit{stable} \\
&(\textit{much easier})
\end{aligned}
$$



13 Newton's Law of Cooling

Newton says that the rate at which heat is lost from an object is proportional to the difference between the temperature of that object and the ambient temperature.

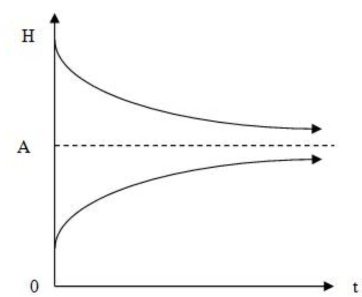
How can we turn this into a differential equation?
Let H=heat
A=ambient temperature

$$\frac{dH}{dt} \quad \underbrace{= \alpha} \quad (A - H)$$

↑ Rate at which heat changes — ↑ is proportional to — ↖ difference between temperature and ambient temperature

The equilibrium is $H = A$ (assuming $\alpha \neq 0$ otherwise objects never lose heat)

$$f(H) = \alpha(A - H)$$
$$f'(H) = -\alpha$$



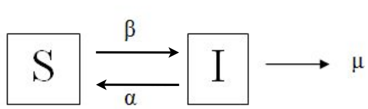
$\therefore\ H = A$ *is stable.*

i.e. eventually objects will warm up to, or cool down to the ambient temperature in the room.

14 Two-Dimensional Differential Equations

SIS epidemic with death:

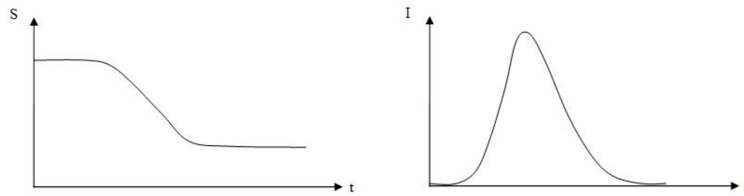
$$S' = \alpha I - \beta SI$$
$$I' = \beta SI - \alpha I - \mu I$$



What can we do with this?
Adding the equations together, we have $S' + I' = -\mu I \neq 0$
Therefore the population is not constant.
The equilibria are:

$$\alpha I - \beta SI = 0 \qquad \Rightarrow I = 0 \ or \ S = \frac{\alpha}{\beta}$$
$$\beta SI - \alpha I - \mu I = 0 \qquad \Rightarrow I = 0 \ or \ S = \frac{\alpha + \mu}{\beta}$$

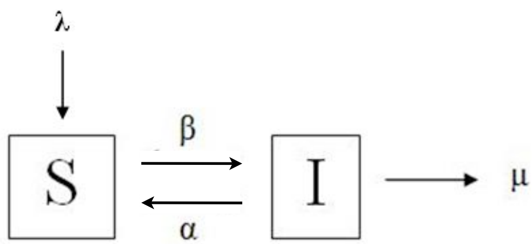
This suggests that eventually $I = 0$. Is this good?
What else? Answer (for now): Numerics

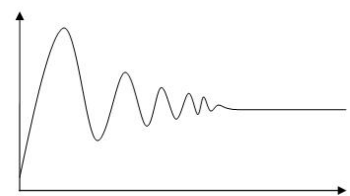


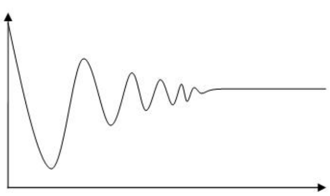
Eventually the disease dies out but it kills a portion of the population along the way.

Example 53. SIS epidemic, with migration

$$
\begin{aligned}
S' &= \lambda + \alpha I - \beta SI \\
I' &= \beta SI - \alpha I - \mu I
\end{aligned}
$$



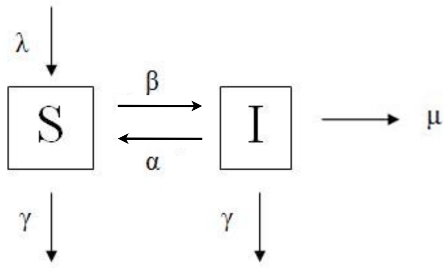


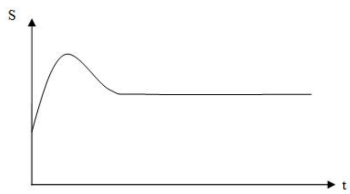


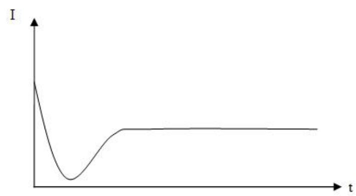
The disease oscillates but settles down.

Example 54. SIS epidemic with migration and natural death

$$
\begin{aligned}
S' &= \lambda + \alpha I - \beta SI - \gamma S \\
I' &= \beta SI - \alpha I - \mu I - \gamma I \\
&\text{Adding together, if } \mu = 0, \text{we have} \\
S' + I' &= \lambda - \gamma(S + I) \\
N' &= \lambda - \gamma N \\
N &\to \frac{\lambda}{\gamma}
\end{aligned}
$$

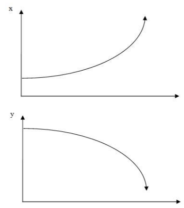




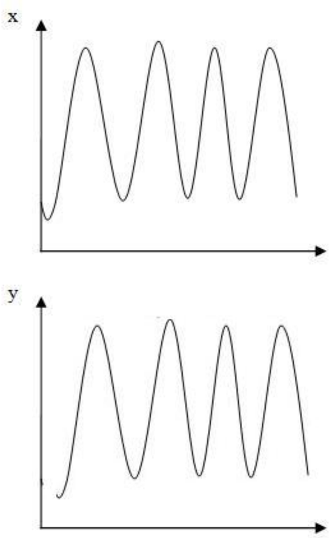


Example 55. Foxes and Rabbits
In the absence of foxes, rabbits grow without bound. In the absence of rabbits, foxes die as there is nothing to eat.
No eating:

$$x' = \lambda x \qquad \text{rabbits grow}$$
$$y' = -\delta y \qquad \text{foxes die}$$



When foxes eat rabbits, rabbits die and foxes increase.
$x' = \lambda x - \epsilon xy$
$y' = \epsilon xy - \delta y$



[The first midterm covers up to here.]