ELG3126x RANDOM SIGNALS AND SYSTEMS Spring 2016

ASSIGNMENT 2 -- Bayes, pmf,

Q1. A company uses three different assembly lines- A_1, A_2, and A_3 to manufacture a particular component. Of those manufactured by line A_1, 4% need rework to remedy a defect, whereas 9% of A_2's components need rework and 11% of A_3's need rework. Suppose that 40% of all components are produced by line A_1, 35% are produced by line A_2, and 25% come from line A_3. If a randomly selected component needs rework, what is the probability that it came from line A_1? From line A_2? From line A_3?

Q1 Answer

$P(A_1) = 0.4$, $P(A_2) = 0.35$, $P(A_3) = 0.25$

$P(R/A_1) = 0.04$, $P(R/A_2) = 0.09$, $P(R/A_3) = 0.11$

TOTAL Probability

$$P(R) = P(R/A_1)P(A_1) + P(R/A_2)P(A_2) + P(R/A_3)P(A_3)$$
$$= 0.04 \times 0.4 + 0.09 \times 0.35 + 0.11 \times 0.25 = \underline{\underline{0.075}}$$

$$P(A_1/R) = \frac{P(R/A_1)P(A_1)}{P(R)} = \frac{0.04 \times 0.4}{0.075} = 0.21$$

$$P(A_2/R) = \frac{P(R/A_2)P(A_2)}{P(R)} = \frac{0.09 \times 0.35}{0.075} = 0.42$$

$$P(A_3/R) = \frac{P(R/A_3)P(A_3)}{P(R)} = \frac{0.11 \times 0.25}{0.075} = 0.37$$

2. A certain city has one morning newspaper and one evening newspaper. It is estimated that 30% of the city's household subscribe to the morning paper and 50% subscribe to the evening paper. Of those who subscribe to the morning paper, 80% also subscribe to the evening paper.
   a) What proportion of households subscribes to both paper?
   b) If a household subscribes to the evening paper, what is probability that they subscribe to the morning paper?
   c) What proportion of households subscribes to at least one paper?
   d) What proportion of households subscribes to the morning paper but not evening paper?
   e) What proportion of households does not subscribe to either paper?
   f) Are the events of subscribing to the morning paper and subscribing to the evening paper independent? Justify your answer.

Q2 Answer

Sol: Define M: morning paper, E: evening paper so $P(M)$ =0.3, $P(E)$ =0.5, $P(E|M) =$ =0.8

a) [4 marks] $P(M \cap E) = P(E|M)P(M)$ = 0.8 x 0.3 = 0.24

b) [4 marks] $P(M|E) = \frac{P(M \cap E)}{P(E)}$ = 0.24/0.5 = 0.48

c) [4 marks] $P(E \cup M) = P(E) + P(M) - P(E \cap M)$ = 0.5 + 0.3 - 0.24 = 0.56

d) [4 marks] $P(M \cap E') = P(M) - P(M \cap E)$ = 0.3 - 0.24 = 0.06

e) [4 marks] $P(M' \cap E') = 1 - P(M \cup E)$ = 1 - 0.56 = 0.44

f) [4 marks] E and M are not independent since $P(E \cap M)$ = 0.24 $\neq P(E)P(M) =$ = 0.5 x 0.3 = 0.15

3. A, B and C are events with probabilities 0.6, 0.2, and 0.7 respectively.
   (a) What are the largest possible values of P(A U B) and (A U C)?
   (b) What are the smallest possible values of P(A U B) and (A U C)?
   (c) What are the largest possible values of P(AB) and P(AC)?
   (d) What are the smallest possible values of P(AB) and P(AC)?

(a) $P(A \cup B) = P(A) + P(B) - P(A \cap B) \leq P(A) + P(B)$. Thus, $P(A \cup B) \leq 0.8$ with equality occurring when $P(A \cap B) = 0$.
Similarly, $P(A \cup C) = P(A) + P(C) - P(A \cap C) \leq P(A) + P(C)$ But, $P(A) + P(C) > 1$ and so we conclude that $P(A \cup C) \leq 1$ with equality occurring when $P(A \cap C) = 0.3$.

(b) Since $A \subset A \cup B$ and $B \subset A \cup B$, we know that $P(A) \leq P(A \cup B)$, $P(B) \leq P(A \cup B)$ and so $\max\{P(A), P(B)\} = 0.6 \leq P(A \cup B)$. Thus, $P(A \cup B) \geq 0.6$ with equality occurring when $B \subset A$.
Similarly, $\max\{P(A), P(C)\} = 0.7 \leq P(A \cup C)$ with equality occurring when $A \subset C$.

(c) Since $A \cap B \subset A$ and $A \cap B \subset B$, we have that $P(A \cap B) \leq P(A)$, $P(A \cap B) \leq P(B)$ and so $P(A \cap B) \leq \min\{P(A), P(B)\} = 0.2$. Thus, $P(A \cap B) \leq 0.2$ with equality occurring when $B \subset A$.
Similarly, $P(A \cap C) \leq \min\{P(A), P(C)\} = 0.6$ with equality occurring when $A \subset C$.

(d) The smallest possible value of $P(A \cap B)$ is 0 when A and B are mutually exclusive. On the other hand, the smallest possible value of $P(A \cap C)$ is 0.3 in which case $P(A \cup C) = 1$ as noted in part (a).
In summary, $0.6 \leq P(A \cup B) \leq 0.8$ where the minimum value occurs when $B \subset A$ (and thus $P(A \cap B) = P(B) = 0.2$ has a maximum value); while the maximum value 0.8 of $P(A \cup B)$ occurs when $P(A \cap B)$ has its minimum possible value 0.
Similarly, $0.6 \leq P(A \cup C) \leq 1$ where the minimum value occurs when $A \subset C$ (and thus $P(A \cap C) = P(A) = 0.6$ has a maximum value); while the maximum value 1 of $P(A \cup C)$ occurs when $P(A \cap C)$ has its minimum possible value 0.3.

Q4.

4. Express each of the following events in terms of the events A, B, and C, and the set operations of complementation, union, and intersection.
   - (a) at least one of the events A, B, C occurs;
   - (b) none of the events A, B, C occurs;
   - (c) all three events A, B, C occur;
   - (d) exactly one of the events A, B, C occurs;
   - (e) at most one of the events A, B, C occurs;
   - (f) events A and B occur, but not C;
   - (g) either event A occurs, or if not then B also does not occur.

Ans A4.

(a) $A \cup B \cup C$.

(b) $A^c B^c C^c$.

(c) ABC.

(d) $(A \cap B^c \cap C^c) \cup (B \cap A^c \cap C^c) \cup (C \cap A^c \cap B^c)$.

(e) $(A^c \cap B^c \cap C^c) \cup (A \cap B^c \cap C^c) \cup (B \cap A^c \cap C^c) \cup (C \cap A^c \cap B^c)$

(f) $A \cap B \cap C^c$.

(g) $A \cup (A^c \cap B^c)$.

Q5.

Three friends (Al, Bob, and Chris) put their names in a hat and each draws a name from the hat. (Assume Al picks first, then Bob, then Chris.)

(a) Find the sample space.

(b) Find the sets A, B, and C that correspond to the events "Al draws his name," "Bob draws his name," and "Chris draws his name."

(c) Find the set corresponding to the event, "no one draws his own name."

(d) Find the set corresponding to the event, "everyone draws his own name."

(e) Find the set corresponding to the event, "one or more draws his own name."

(g) Find $P(B \cap C / A)$ and $P(C / A \cap B)$

Soln A5 a)-e)

(a) $S = \{abc,\ cab,\ bca,\ acb,\ bac,\ cba\}$

(b) $A = \{abc, acb\}$ $B = \{abc, cba\}$ $C = \{abc, bac\}$

(c) $(A \cup B \cup C)^c = \{abc, acb, cba, bac\}^c = \{cab,\ bca\}$

(d) $A \cap B \cap C = \{abc\}$

(e) $A \cup B \cup C = \{abc, acb, cba, bac\}$.

SOL'n A5g)

$P[B \cap C | A] = P[$ Bob & Chris pick their names / Al picked his name $]$

$$= \frac{P[B \cap C \cap A]}{P[A]} = \frac{P[\{abc\}]}{P[\{abc, acb\}]} = \frac{1/6}{2/6} = \frac{1}{2}.$$

$P[C | A \cap B] = P[$ Chris picks his name | Al & Bob picked their name $]$

$$= \frac{P[A \cap B \cap C]}{P[A \cap B]} = \frac{P[\{abc\}]}{P[\{abc\}]} = 1.$$

Q6.

A number x is selected at random in the interval $[-1, 2]$. Let the events $A = \{x < 0\}$, $B = \{|x - 0.5| < 0.5\}$, and $C = \{x > 0.75\}$. Find $P[A|B]$, $P[B|C]$, $P[A|C^c]$, $P[B|C^c]$.

Answer A6

A= [-1, 0), B= (0, 1), and C= (.75, 2] so (AB) = 0, (BC)= (.75, 1), and (AC)= 0.

Also AC'= [-1,0), BC'=(0, .75]

P[A]=[-1,0)/[-1,2]=1/3, similarly P[B] = 1/3, P[C]= 1.25/3 = 5/12, P[C']=1-5/12=7/12

So P[A/B] = P[AB]/P[B] = 0/.333 = 0

$$P[B|C] = \frac{P[B \cap C]}{P[C]} = \frac{P[(0.75, 1)]}{P[(0.75, 2)]} = \frac{1/12}{5/12} = \frac{1}{5}$$

$$P[A|C^c] = \frac{P[A \cap C^c]}{P[C^c]} = \frac{P[(-1,0)]}{P[[-1, 0.75]]} = \frac{1/3}{7/12} = \frac{4}{7}$$

P[B/C'] = P[BC']/P[C'] = P[(0,.75]]/(7/12) = (3/12)/(7/12) = 3/7

Q7. A ternary communication system is shown in Fig. P2.4. Suppose that input symbols 0, 1, and 2 occur with probability 1/3 respectively.

(a) Find the probabilities of the output symbols.

(b) Suppose that a 1 is observed at the output. What is the probability that the input was 0? 1? 2?

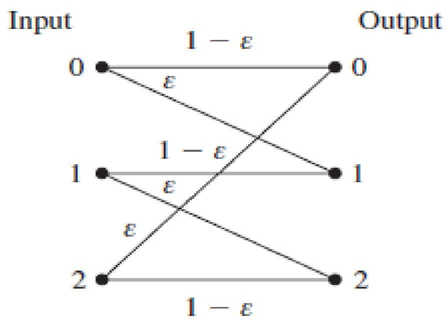

FIGURE P2.4

Q 8.

In the ternary communication system in Problem 2.81, is there a choice of ε for which the input of the channel is independent of the output of the channel?

A7.

Let X denote the input and Y the output.

a) $P[Y=0] = P[Y=0|X=0]P[X=0] + P[Y=0|X=1]P[X=0] + P[Y=0|X=2]P[X=2]$

$= [illegible] = (1-\varepsilon)\frac{1}{3} + \varepsilon\frac{1}{3} = \frac{1}{3}$

larly

$$P[Y=1] = [illegible] \ \frac{1}{3}$$

$$P[Y=2] = [illegible] \ \frac{1}{3}$$

b) Using Bayes' Rule

$$P[X=0|Y=1] = \frac{P[Y=1|X=0]P[X=0]}{P[Y=1]} = \frac{\frac{1}{3}\varepsilon}{\frac{1}{3}} = \varepsilon$$

$$P[X=1|Y=1] = \frac{P[Y=1|X=1]P[X=1]}{P[Y=1]} = \frac{(1-\varepsilon)\frac{1}{3}}{\frac{1}{3}} = 1-\varepsilon$$

$$P[X=2|Y=1] = 0$$

A8.

Regardless of the value of ϵ, we always have

$P[X=2 \mid Y=1] = 0 \neq P[X=2] = \frac{1}{3}$

$\therefore$ the output cannot be independent of the input.

Q 9.

Let U be selected at random from the unit interval. Let $A = \{0 < U < 1/2\}$, $B = \{1/4 < U < 3/4\}$, and $C = \{1/2 < U < 1\}$. Are any of these events independent?

A 9

$P[A \cap B] = P[\frac{1}{4} < U < \frac{1}{2}] = \frac{1}{4} = P[A]P[B] = \frac{1}{2}\frac{1}{2}$ ✓ A & B indep

$P[A \cap C] = 0 \neq P[A]P[C] = \frac{1}{2}\frac{1}{2} \Rightarrow$ A & C Not Indep.

$P[B \cap C] = P[\frac{1}{2} < U < \frac{3}{4}] = \frac{1}{4} = P[B]P[C] = \frac{1}{2} \cdot \frac{1}{2}$ ✓ B & C indep.

Q 10.

Alice and Mary practice free throws at the basketball court after school. Alice makes free throws with probability p_a and Mary makes them with probability p_m. Find the probability of the following outcomes when Alice and Mary each take one shot: Alice scores a basket; Either Alice or Mary scores a basket; both score; both miss.

A10

Let $A = \{$Alice makes shot$\}$ $M = \{$Mary makes shot$\}$

We assume that A and M are independent

$P[A] = p_a$

since $AM^c \cap A^cM = \phi$

$P[\text{one makes a shot}] = P[AM^c \cup A^cM] = P[AM^c] + P[A^cM]$

$= p_a(1-p_m) + (1-p_a)p_m$ by independence

$P[AM] = p_a p_m$

$P[A^cM^c] = (1-p_a)(1-p_m)$.

Q 11. Let A, B, and C be events with probabilities $P[A]$, $P[B]$, and $P[C]$.

(a) Find $P[A \cup B]$ if A and B are independent.

(b) Find $P[A \cup B]$ if A and B are mutually exclusive.

(c) Find $P[A \cup B \cup C]$ if A, B, and C are independent.

(d) Find $P[A \cup B \cup C]$ if A, B, and C are pairwise mutually exclusive.

A 11.

(a) $P[A \cup B] = P[A] + P[B] - P[A \cap B] = P_A + P_B - P_A P_B$

(b) $P[A \cup B] = P[A] + P[B] = P_A + P_B$

(c) $P[A \cup B \cup C] = P[A] + P[B] + P[C] - P[A \cap B] - P[A \cap C] - P[B \cap C] + P[A \cap B \cap C]$

$= P_A + P_B + P_C - P_A P_B - P_A P_C - P_B P_C + P_A P_B P_C$

(d) $P[A \cup B \cup C] = P_A + P_B + P_C$

Q 12. An information source produces binary triplets $\{000, 111, 010, 101, 001, 110, 100, 011\}$ with corresponding probabilities $\{1/4, 1/4, 1/8, 1/8, 1/16, 1/16, 1/16, 1/16\}$. A binary code assigns a codeword of length $-\log_2 p_k$ to triplet k. Let X be the length of the string assigned to the output of the information source.

(a) Show the mapping from S to S_X, the range of X.

(b) Find the probabilities for the various values of X.

A 12.

$$
\begin{array}{lcccccccc}
S = \{ & 000, & 111, & 010, & 101, & 001, & 110, & 100, & 011 \} \\
 & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
X(\zeta): & 2 & 2 & 3 & 3 & 4 & 4 & 4 & 4
\end{array}
$$

$P[X=2] = P[\{000, 111\}] = \frac{1}{2}$

$P[X=3] = P[\{010, 101\}] = \frac{1}{4}$

$P[X=4] = P[\{001, 110, 100, 011\}] = \frac{1}{4}$