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ELG 3126 **RANDOM SIGNALS AND SYSTEMS,** J. Lostracco, SITE 0110B, jlostrac@uottawa.ca
Assignment 1: Set Theory Review, Due Weds 11 May

1. The pizza pizza phone number 613-737-1111 has k distinct digits in it. State the set of these digits and all the subsets of this set that have one or two elements.

Answer 1: S={1,3,6,7}, so for 1 element subsets, E1= {1},{3},{6},{7} (4 subsets of 1 element)
and for 2 element subsets, E2= {1,3}, {1,6}, {1,7}, {3,6}, {3,7}, {6,7}. (6 subsets of 2 elements).

- In general, for a set of n objects, there are C(n,k) subsets of size k, so there are n subsets of size 1 and 0.5n(n-1) subsets of size 2.
  - In Q1, for 1 element; n=4, k=1 so C(4,1)=4!/((4-1)!*1!) = 4 combinations of 1 element
  - for 2 elements; n=4, k=2 so C(4,2)=4!/((4-2)!*2!) = 6 combinations of 2 elements.

2. Suppose four teams, numbered one through four, play a round-robin tournament: every pair of teams play a game. Two teams play each game and one of them wins; ties do not occur.

(a) Define a set W so that the elements of W correspond to the possible scenarios of the tournament. An element of W should specify the entire sequence of outcomes of the games. Explain how the elements of your set correspond to the possible scenarios.
(b) How many possible scenarios are there?

2.

(a) There are six games in all. We could order the games in any convenient way; for instance, the first game could be between teams 1 and 2, the second between teams 1 and 3, the third between teams 1 and 4, the fourth between teams 2 and 3, the fifth between 2 and 4 and the sixth between teams 3 and 4. Then one choice would be $\Omega = \{(w_1, w_2, w_3, w_4, w_5, w_6) : w_k \in \{0, 1\}\}$. Here w_k represents a loss (0) or a win (1) for the lower numbered team in game numbered k, for each $k = 1 \ldots 6$.

(b) $2^6 = 64$, because there are two possible outcomes for each of the six games.

3. Suppose A and B are events for some probability space such that P(AB) = 0.2 and P(A U B) = 0.5. Find the set of possible values of the pair (P(A), P(B)) and sketch this set.

3.

(This is one of many ways to get the answer.) Since, by De Morgan's law, the complement of $A \cup B$ is A^cB^c, the fact $P(A \cup B) = 0.5$ is equivalent to $P(A^cB^c) = 0.5$. Thus, we can fill in the Karnaugh diagram for A and B as shown:

	B^c	B
A^c	0.5	$0.3-a$
A	a	0.2

We filled in the variable a for $P(AB^c)$, and then, since the sum of the probabilities is one, it must be that $P(A^cB) = 0.3 - a$. The valid values of a are $0 \le a \le 0.3$, and $(P(A), P(B)) = (0.2 + a, 0.5 - a)$. So, in parametric form, the set of possible values of $(P(A), P(B))$ is $\{(0.2 + a, 0.5 - a) : 0 \le a \le 0.3\}$. Equivalent ways to write this set are $\{(u, v) : v = 0.7 - u \text{ and } 0.2 \le u \le 0.5\}$ or $\{(x, 0.7 - x) : 0.2 \le x \le 0.5\}$.

4. Suppose twelve students in a class are to be grouped into teams.
   (a) If each team has three students, how many ways are there to form teams? (The ordering of students within teams does not matter, and the ordering of the teams does not matter.)
   (b) If each team has either two or three students, how many ways are there to form teams?

(a) One solution is the following. Four teams, numbered one through four, can be sequentially selected as follows. To begin, there are $\binom{12}{3}$ ways to choose team one. That leaves ine students, so for any choice of team one, there are $\binom{9}{3}$ ways to choose team two. That leaves $\binom{6}{3}$ ways to choose team three, and $\binom{3}{3} = 1$ way to choose team four. Thus, there are $\binom{12}{3}\binom{9}{3}\binom{6}{3}\binom{4}{2}\binom{3}{3}$ ways to choose teams numbered one through five. But team numbers don't matter, and there are 4! ways to number the teams. So each team formation can be arrived at by the above procedure 4! ways. So the number of ways to form teams (with team numbers not mattering) is $\frac{\binom{12}{3}\binom{9}{3}\binom{6}{3}\binom{3}{3}}{4!} = 15400$.
A second solution is the following. There are 12! orderings of the students in the class. Given an ordering, we can pair the first student with the second and third to make the first team, pair the fourth with the fifth and sixth to make the second team, and so on. Because the ordering within teams and the ordering of the teams themselves does not matter, many permutations produce an equivalent formation of teams. We need to count, for one formation of teams, how many permutations produce that formation. The number is $6^4 4!$, because there are six ways to order the students within each team, and 4! ways to order the teams. So the total number of team formations with two students per team is 15400.

(b) As already found in part (a), there are 15400 ways to form teams with all teams of size three. So the only other possibility for team sizes are six tams of size two and two teams of size three and three teams of size two.
The first case is very similar to that in part (a). The number of teams is simply $\frac{\binom{12}{2}\binom{10}{2}\binom{8}{2}\binom{6}{2}\binom{4}{2}\binom{2}{2}}{6!} =$ 10395. You can also check your answer the second way: $\frac{12!}{2^6 6!} = 10395$.
Now for the second case. Five teams, numbered one through five, such that teams one and two each have three students and teams three, four and five each have two students, can be formed as follows. To begin, there are $\binom{12}{3}$ ways to choose team one. That leaves nine students, so for any choice of team one, there are $\binom{9}{3}$ ways to choose team two. That leaves $\binom{6}{2}$ ways to choose team three, and $\binom{4}{2}$ ways to choose team four and $\binom{2}{2}$ ways to choose team five. Thus, there are $\binom{12}{3}\binom{9}{3}\binom{6}{2}\binom{4}{2}\binom{2}{2}$ ways to choose teams numbered one through five, such that teams one and two have three students each and teams three and four and five have two students each. But team numbers don't matter, so for a given formation of teams with two of size three and two of size two , there are two ways to decide which would be team one and which would be team two, and two ways to decide which would be team three and which would be team four. So each team formation can be arrived at by the above procedure in $2 \cot 6 = 12$ ways. So the number of ways to form teams (with team numbers not mattering) is $\frac{\binom{12}{3}\binom{9}{3}\binom{6}{2}\binom{4}{2}\binom{2}{2}}{2 \cdot 6}$. This number simplifies to $\frac{10!}{(3!)^2 2^2 2^2} = 138,600$.
Therefore, the total number of team formations with each team having either two or three students is $10395 + 138600 + 15400 = 164395$.

5. Suppose five cards are drawn from a standard 52 card deck of playing cards, with all possibilities being equally likely.

(a) E1 is the event that all five cards are one of A, K, Q, (though not necessarily of the same suit). Find P(E1):

(b) E2 is the event that either all the five cards are from the same suit or ~~four~~ **three** of the five cards are A, K, Q, from a common suit. Find P(E2):

A5.

a) There are 3x4=12 cards in the deck that are A,K,Q. Thus there are C(12,5) = 12!/(7!x5!) =12x11x10x9x8/(5x4x3x2) = 792 ways of choosing cards for E1. Normalizing by the number of all outcomes, C(52, 5), we get P(E1)= C(12,5)/C(52/5)=792/2598960 = 0.0003

b) E2 is the union of two events E4 and E3 which are defined as;

- **E3 is the event of having 3 of the cards A, K, Q of the same suit.**
- **E4 is the event of having all 5 cards of the same suit.**

The number of outcomes in E3 is 4 x 49 x48 = 9408 because there are four suits to choose from and the 4^{th} and 5^{th} cards can be any of the remaining 49 and 48 cards.

The number of outcomes in E4 is C(13,5)x4 =

The number of outcomes common to E1 and E3 is 4x(9 x 8) = since there are 4 suits and 9 remaining A,K,Q.

So number of outcomes in E2 = E3 U E4 = 9408 + 5148 – 288 = 14268.

So P(E2)= 14268/C(52,5) =14268/2598960 = 0.0055

6. The results of three flips of a biased coin are observed. The probability of a head occurring is 0.4. Consider the following events:

A: At least two heads are observed

B: Exactly one tail is observed

C: Exactly two tails are observed

a) List an appropriate sample space S for this experiment.

b) Are the events of getting a head on a flip independent for the 3 flips?

c) Find P(A); P(B'); P(B ∩ C); P(A ∩ B); P(B /A); P(A U B)

d) Is there a pair of mutually exclusive or independent events among A, B and C? If so which pairs are mutually exclusive or independent?

Answer Q6.

The results of three flips a **biased** coin are observed. The probability of a head occurring is 0.4. Consider the following events:

A: At least two heads are observed B: Exactly one tail is observed

C: Exactly two tails are observed

a) [*1 marks*] List an appropriate sample space S for this experiment.

Sol:

S={(HHH),(HHT),(HTH),(THH),(TTH),(THT),(HTT),(TTT)}

b) [*1 marks*] Are the events of getting a head on flip independent for the 3 flips?

Sol:

Yes, event of a head on one flip **is independent** of event of a head on any other flip.

c) [*3 marks*] Find $P(A), P(B'),\ P(B \cap C),\ P(A \cap B),\ P(B|A),\ P(A \cup B)$

Sol:

$A = \{(HHT),(HTH),(THH),(HHH)\} \rightarrow P(A) = 3(0.4 \times 0.4 \times 0.6) + (0.4 \times 0.4 \times 0.4)$
$= 0.352$

$B = \{(HHT),(HTH),(THH)\} \rightarrow P(B) = 3(0.4 \times 0.4 \times 0.6) = 0.288$

$C = \{(HTT),(THT),(TTH)\} \rightarrow P(C) = 3(0.6 \times 0.6 \times 0.4) = 0.432$

$B \cap C = \varnothing \rightarrow P(B \cap C) = 0,$

$A \cap B = B \rightarrow P(A \cap B) = P(B) = 3(0.4 \times 0.4 \times 0.6) = 0.288$

$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{0.288}{0.352} = 0.82$

$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.352 = P(A)$

d) [*2 marks*] Is there a pair of mutually exclusive or independent events among A, B and C? If so which pairs are mutually exclusive or independent?

Sol:

A and C are M.E and also B and C are M.E and they are not independent. A and B are not independent and are not M.E, since $P(B|A) = 0.82 \neq P(B)$.

7. A student is preparing for an exam by studying a list of 10 problems. She can solve 5 of them. For the exam, the instructor selects 4 questions at random from the list of 10. What is the probability that the student can solve all 4 problems on the exam?

A7. N=10, M=5, n=4, X= # of correct answers has a hypergeometric distribution.
P(X=4) = C(5,4)C(6,0)/C(10,4) = 5/(10x9x8x7x0.5) =5/2520= 0.00198

8. If S = {2,3,4,6,10}, A = {4, 6} and B = {3, 4, 6, }, determine (i) A', (ii) A ∩ B, (iii) B−A, (iv) A−B, and (v) A' ∩ B.

Answ 8: If S = {2,3,4,6,10}, A = {4, 6} and B = {3, 4, 6,}, determine (i) A', (ii) A ∩ B, (iii) B−A, (iv) A−B, and (v) A' ∩ B.
• (i) A' = [2,3,10], (ii) AB= [4,6], (iii) B-A=3 (iv) A-B= [null set] (v) A'B= [3]

9. Determine whether the following sets are finite, infinitely countable or uncountable:

A = {null, red, green, 1.333},
B = {x | x^2 − 2x + 1 = 0, x € R},
C = N_0 = {0, 1, 2, . . .},
D = {students currently attending kindergarten worldwide},
E = {Canadian teams in the NHL playoffs},
F = {students currently enrolled in a mining engineering program at the University of Ottawa},
G = {x | x € R, 0 < x < 99} = (0, 99),
H = {x | x^2 + 3x + 2 = 0, x € I},
I = {−1,−2} ∩ [0, 3].

Determine if any of these sets is equal to any other; and which is a subset of another?

Answ 9:
A is a finite set of 4 elements
B={1} since x^2-2x+1=0 has only one root, so B is a finite set of one element.
C is a countably infinite set.
D is a countably infinite set.
E and F are finite sets (null sets) since there are no Canadian teams in the NHL playoffs this year and there is no mining engineering program at uOttawa. (only data mining – minerals and earth resources part of Geology)
G in an uncountable set since there are an infinite number of real numbers between 0 and 99.
H={-1,-2} so is a finite countable set.
I is a null set. There is no intersection of –ve integers and real positive number less than or equal to 3.

Since E, F and I are a null sets, they are subsets of all others. Otherwise B is a subset of C, D and G.