NAME: ______________________ STUDENT ID# : ______________

LAKEHEAD UNIVERSITY

FACULTY OF ENGINEERING

Final Exam

NASER Solutions

Engineering	3021	AA	Summer Transition Program
SUBJECT	COURSE NO.	SECTION	

Engineering Analysis A	Drs. H. Naser and M. N. Uddin
COURSE TITLE	INSTRUCTOR

Thu, Aug. 6, 2015	9:00 a.m. - 12:00 p.m.	3 hours	UC2001/AT1003
EXAM DATE	EXAM TIME	DURATION	ROOM

- NO CALCULATOR OR ELECTRONIC DEVICES ALLOWED
- SHOW YOUR STEPS – NO STEPS NO MARKS
- CLOSED BOOK EXAM

This exam **MAY NOT** be removed from the room

STUDENTS PLEASE NOTE

YOU MUST count the number of pages (11) in this examination question paper **BEFORE** beginning to write, and report any discrepancy immediately to a proctor.

This section is for faculty use only

Q	1	2.a	2.b	3	4	5	6	7	8	9	10	Total
M	10	5	5	10	10	10	10	10	10	10	10	100

1. Discuss the curve $y = f(x)$ with respect to local maxima and minima, concavity, points of inflection, and odd/even symmetry. Use this information to sketch the curve. 10

$$f(x) = x + \frac{4}{x}$$

$f'(x) = 1 - \frac{4}{x^2} = \frac{x^2-4}{x^2} = \frac{(x-2)(x+2)}{x^2}$

$f'(x) = 0 \Rightarrow x = 2,\ x = -2$

$f'(x)$ does not exist if $x = 0$

Hence, critical numbers of f : $x = -2, 0, 2$ 3

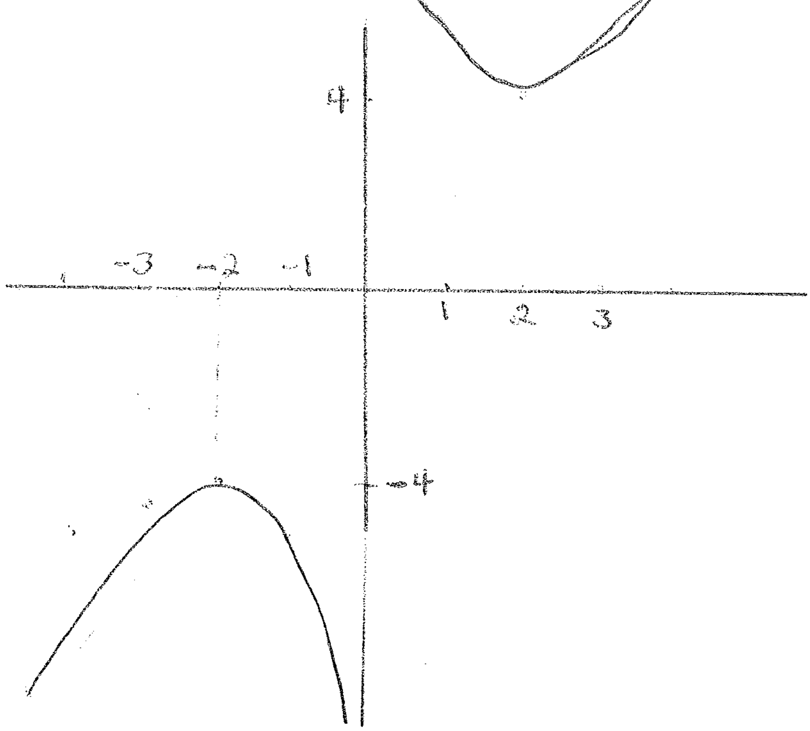
$f''(x) = 8x^{-3} = \frac{8}{x^3}$ $f''(x)$ does not exist if $x = 0$ 1

	$-\infty$	-2		0		2	$+\infty$
x^2-4	+	0	−		−	0	+
x^2	+		+	0	+		+
$f'(x)$	+	0	−	∞	−	0	+
$f(x)$	↗	local max	↘		↘	local min	↗
$f''(x)$	−		−	∞	+		+
Concavity Test	C.D		C.D.	infl. pt.	C.U		C.U

3

$f(-x) = (-x) + \frac{4}{(-x)} = -\left[x + \frac{4}{x}\right] = -f(x) \Rightarrow f$ is odd

Hence : reflection about the origin. 1



2

2. (a) Find $\lim_{x\to 0}\dfrac{\ln(1+x^2)}{e^x-\cos x}$. Indeterminate type $\frac{0}{0}$ (5)

L'Hospital rule:

$$= \lim_{x\to 0} \frac{\frac{1}{1+x^2}\cdot(2x)}{e^x+\sin x} = \lim_{x\to 0}\frac{2x}{(1+x^2)(e^x+\sin x)} = \frac{0}{(1+0)(e^0+0)} = 0$$

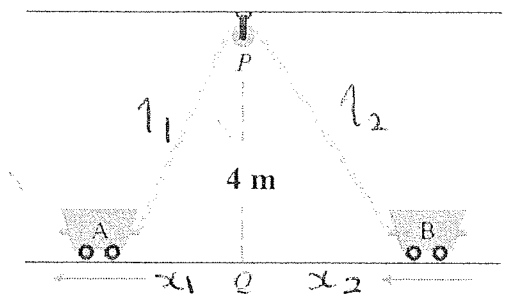
(b) $\lim_{x\to(\pi/2)^-} \sec x - \tan x$ indeterminate type $\infty - \infty$ (5)

$$\lim_{x\to\frac{\pi}{2}^-} \sec x - \tan x = \lim_{x\to\frac{\pi}{2}^-} \frac{1}{\cos x} - \frac{\sin x}{\cos x} = \lim_{x\to\frac{\pi}{2}^-} \frac{1-\sin x}{\cos x}$$ indeterminate type $\frac{0}{0}$

Apply Hospital rule:

$$\lim_{x\to\frac{\pi}{2}^-}\frac{1-\sin x}{\cos x} = \lim_{x\to\frac{\pi}{2}^-}\frac{-\cos x}{-\sin x} = \lim_{x\to\frac{\pi}{2}^-}\frac{\cos x}{\sin x} = 0$$

3. Two carts, A and B, are connected by a rope 12 m long that passes over a pulley P. The point Q is on the floor 4 m directly beneath P and between the carts. Cart A is being pulled away from Q at a speed of 0.5 m/s. How fast is cart B moving toward Q at the instant when cart A is 3 m from Q? 10



Hint: Let x_1 and x_2 be the distances from cart A to Q and from cart B to Q at any point in time. Use the right triangles PAQ and PBQ to relate x_1 and x_2.

Given : $l_1 + l_2 = 12$ m

$\frac{dx_1}{dt} = 0.5$ m/s

(We chose Q to be the origin, and We chose +ve direction of x_1 and x_2 to be from right to left). 1

Unknown : $\left.\frac{dx_2}{dt}\right|_{x_1 = 3\text{ m}} = ?$

Using Pythagorean theorem

$l_1 = \sqrt{x_1^2+16}$ and $l_2 = \sqrt{x_2^2+16}$ 3

Hence :

$\sqrt{x_1^2+16} + \sqrt{x_2^2+16} = 12$

Using chain rule to differentiate both side w.r.t. t :

$\frac{1}{2}(x_1^2+16)^{-\frac{1}{2}}(2x_1)\frac{dx_1}{dt} + \frac{1}{2}(x_2^2+16)^{-\frac{1}{2}}(2x_2)\frac{dx_2}{dt} = 0$ 3

$\Rightarrow \frac{dx_2}{dt} = \frac{-x_1/\sqrt{x_1^2+16}}{x_2/\sqrt{x_2^2+16}} \cdot \frac{dx_1}{dt} = \frac{-x_1\sqrt{x_2^2+16}}{x_2\sqrt{x_1^2+16}} \cdot \frac{dx_1}{dt}$

when $x_1 = 3$ m $\Rightarrow \sqrt{3^2+16} + \sqrt{x_2^2+16} = 12$

$\Rightarrow \sqrt{25} + \sqrt{x_2^2+16} = 12$

$\Rightarrow \sqrt{x_2^2+16} = 7$ (1)

$\Rightarrow x_2^2+16 = 49$

$\Rightarrow x_2 = \pm\sqrt{33}$

$\Rightarrow x_2 = -\sqrt{33}$

$$\therefore \frac{dx_2}{dt} = -\frac{3 \cdot \sqrt{33+16}}{(-\sqrt{33})\sqrt{9+16}} \cdot (0.5)$$

$$= \frac{21\,(0.5)}{\sqrt{33}\cdot 5} = \frac{21}{10\sqrt{33}} \simeq 0.37 \text{ m/s.}$$ (2)

4. Evaluate the integral. 10

$$\int \frac{\cos x}{\sin^2 x - \sin x - 6}\,dx$$

[Hint: use substitution technique followed by partial fractions technique.]

$u = \sin x \Rightarrow du = \cos x\,dx$

$$\int \frac{\cos x\,dx}{\sin^2 x - \sin x - 6}\,dx = \int \frac{du}{u^2 - u - 6}$$ 4

$$\frac{1}{u^2-u-6} = \frac{1}{(u-3)(u+2)} = \frac{A}{u-3} + \frac{B}{u+2}$$

$1 = A(u+2) + B(u-3)$

let $u = -2 \Rightarrow 1 = B(-5) \Rightarrow B = -\frac{1}{5}$

let $u = 3 \Rightarrow 1 = A(5) \Rightarrow A = \frac{1}{5}$ 3

$$\frac{1}{u^2-u-6} = \frac{1/5}{u-3} - \frac{1/5}{u+2}$$

$$\int \frac{du}{u^2-u-6} = \frac{1}{5}\ln|u-3| - \frac{1}{5}\ln|u+2| + C$$

$\therefore$ Integral $= \frac{1}{5}\ln|\sin x - 3| - \frac{1}{5}\ln|\sin x + 2| + C$ 3

5. Use integration by parts to evaluate the integral. (10)

$$\int t(\ln t)^2 dt$$

$u=(\ln t)^2 \Rightarrow du = 2(\ln t)\cdot\frac{1}{t}dt = \frac{2\ln t}{t}dt$

$dv = t\,dt \Rightarrow v=\frac{1}{2}t^2$

$$\int t(\ln t)^2 dt = \left(\frac{1}{2}t^2\right)(\ln t)^2 - \int\left(\frac{1}{2}t^2\right)\left(\frac{2\ln t}{t}\right)dt$$

$$= \frac{1}{2}t^2(\ln t)^2 - \int t\ln t\,dt$$

(5)

Apply by-Parts again:

$u=\ln t \Rightarrow du = \frac{dt}{t}$

$dv = t\,dt \Rightarrow v = \frac{1}{2}t^2$

$$\Rightarrow \int t\ln t\,dt = \left(\frac{1}{2}t^2\right)(\ln t) - \int\frac{1}{2}t^2\cdot\frac{1}{t}dt$$

$$= \frac{1}{2}t^2\ln t - \frac{1}{2}\int t\,dt = \frac{1}{2}t^2\ln t - \frac{1}{4}t^2$$

$$\therefore \int t(\ln t)^2 dt = \frac{1}{2}t^2(\ln t)^2 - \left[\frac{1}{2}t^2\ln t - \frac{1}{4}t^2\right] + C$$

$$= \frac{1}{2}t^2(\ln t)^2 - \frac{1}{2}t^2\ln t + \frac{1}{4}t^2 + C$$

$$= \frac{1}{2}t^2\left[(\ln t)^2 - (\ln t) + \frac{1}{2}\right]$$

(5)

6. Find a polar equation for the given Cartesian equation.

10

$$x^3 + y^3 = 3xy$$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

3

$$\Rightarrow (r\cos\theta)^3 + (r\sin\theta)^3 = 3(r\cos\theta)(r\sin\theta)$$

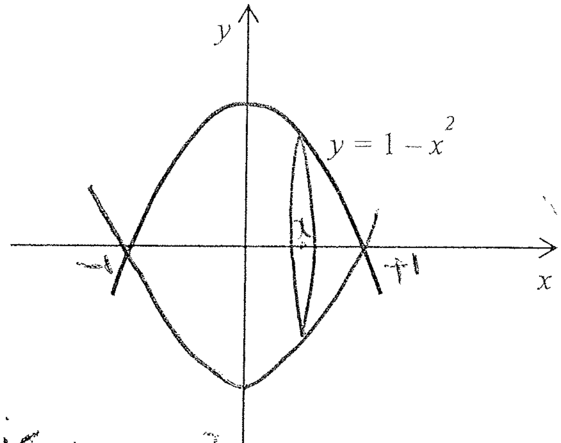
$$\Rightarrow r^3\left[\cos^3\theta + \sin^3\theta\right] = 3r^2\sin\theta\cos\theta$$

$$\Rightarrow r = \frac{3\sin\theta\cos\theta}{\sin^3\theta + \cos^3\theta}$$

7

7. Find the volume of the solid that is generated by rotating around the x-axis the region bounded by the curve $y = 1 - x^2$ and the x-axis. 10

$(V = \int_a^b A(x)dx)$.



The cross-section is a circle with radius $1-x^2$

$\Rightarrow A(x) = \pi(1-x^2)^2$ 5

Points of x-crossings : $1-x^2=0 \Rightarrow x = \pm 1$

$$V = \int_{-1}^{1} \pi(1-x^2)^2 dx = \int_{-1}^{1} \pi[1-2x^2+x^4]dx$$

$$= \pi\left[x - \frac{2}{3}x^3 + \frac{1}{5}x^5\right]_{-1}^{1}$$

$$= \pi\left[1-\frac{2}{3}+\frac{1}{5}\right] - \pi\left[-1+\frac{2}{3}-\frac{1}{5}\right]$$

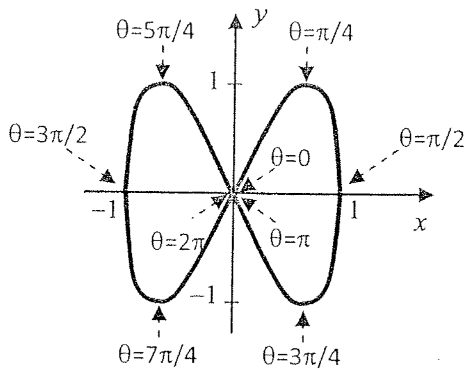
$$= 2\cdot\pi\left(1-\frac{2}{3}+\frac{1}{5}\right)$$

$$= \frac{16\pi}{15}$$ 5

8. The curve below is given by the parametric equations: 10

$$x = \sin\theta, \qquad y = \sin 2\theta$$

a) Label the curve with the values of the parameter θ, $0 \le \theta \le 2\pi$.
b) Then find the equation of the tangent line to the curve at $\theta = \pi/4$.



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$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{2\cos 2\theta}{\cos\theta}$$

$$\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{4}} = \frac{2\cos(\pi/2)}{\cos(\pi/4)} = 0$$

4

At $\theta = \frac{\pi}{4}$ $\quad x = \frac{1}{\sqrt{2}}$, $y = 1$

$\Rightarrow \quad y - 1 = 0\left(x - \frac{\pi}{4}\right)$

$\Rightarrow \quad y = 1$

3

9. Find the radius of convergence and interval of convergence of the following series. 10

$$\sum_{n=1}^{\infty} \frac{2^n (x-3)^n}{n^2}$$

[Ratio Test: $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$]

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2^{n+1}(x-3)^{n+1}}{(n+1)^2} \cdot \frac{n^2}{2^n(x-3)^n}\right| \quad 2$$

$$= \lim_{n\to\infty}\left|2(x-3)\cdot\frac{n^2}{(n+1)^2}\right| = 2|x-3| \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^2 = 2|x-3|$$

$$2|x-3| < 1 \quad 2 \quad \Rightarrow |x-3| < \frac{1}{2} \Rightarrow \frac{-1}{2} < x-3 < \frac{1}{2} \Rightarrow \frac{5}{2} < x < \frac{7}{2} \quad 1$$

Radius of conv. $R = \frac{1}{2}$

2

At $x = \frac{5}{2} \Rightarrow \sum_{n=1}^{\infty} \frac{2^n\left(\frac{-1}{2}\right)^n}{n^2} = \sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{n^2}$

Using alternating series test: $b_n = \frac{1}{n^2}$

i) $b_{n+1} \le b_n \Rightarrow \frac{1}{(n+1)^2} \le \frac{1}{n^2} \Rightarrow (n+1)^2 \ge n^2 \Rightarrow n+1 \ge n$ true for all $n > 0$. ✓

ii) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n^2} = 0$ ✓

∴ the series converges at $x = \frac{5}{2}$

2

At $x = \frac{7}{2} \Rightarrow \sum_{n=1}^{\infty} \frac{2^n\left(\frac{1}{2}\right)^n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$ convergent p-series with $p = 2 > 1$

∴ the series converges at $x = \frac{7}{2}$

1 $\Rightarrow$ Interval of Conv. $= \left[\frac{5}{2}, \frac{7}{2}\right]$.

10. Find the Taylor series for $f(x)=\dfrac{1}{\sqrt{x}}$ centered at $a=1$. 10

[Taylor series: $f(x)=\sum_{n=0}^{\infty}\dfrac{f^{(n)}(a)}{n!}(x-a)^n$]

5

$$f(x)=\frac{1}{\sqrt{x}}=x^{\frac{-1}{2}} \Rightarrow f(1)=1 \qquad 1$$

$$f'(x)=\left(\frac{-1}{2}\right)x^{\frac{-3}{2}}=\frac{-1}{2\sqrt{x^3}} \Rightarrow f^{(1)}(1)=\left(\frac{-1}{2}\right) \qquad \frac{-1}{2}$$

$$f^{(2)}(x)=\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)x^{-5/2}=\frac{3}{2^2\sqrt{x^5}} \Rightarrow f^{(2)}(1)=\frac{1\cdot3}{2^2} \qquad \frac{3}{4}$$

$$f^{(3)}(x)=\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\left(\frac{-5}{2}\right)x^{-7/2}=-\frac{3\cdot5}{2^3\sqrt{x^7}} \Rightarrow f^{(3)}(1)=-\frac{1\cdot3\cdot5}{2^3} \qquad \frac{15}{8}$$

$$\frac{105}{16}$$

$$\vdots$$

$$f^{(n)}(x)=\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\left(\frac{-5}{2}\right)\cdots\left(-\frac{2n-1}{2}\right)x^{-\frac{2n+1}{2}} \Rightarrow \underbrace{f^{(n)}(1)=(-1)^n\frac{1\cdot3\cdot5\cdot7\cdot\ldots\cdot(2n-1)}{2^n}}_{2},\quad n\geq 1$$

3

$$\Rightarrow \frac{1}{\sqrt{x}}=1+\sum_{n=1}^{\infty}(-1)^n\frac{1\cdot3\cdot5\cdot\ldots(2n-1)}{2^n\cdot n!}(x-1)^n$$

$$=1-\frac{1}{2\cdot1!}(x-1)+\frac{1\cdot3}{2^2\cdot2!}(x-1)^2+\frac{1\cdot3\cdot5}{2^3\cdot3!}(x-1)^3+\cdots$$

$$+(-1)^n\frac{1\cdot3\cdot5\cdot\ldots\cdot(2n-1)}{2^n\cdot n!}(x-1)^n+\cdots$$

$$1-\frac{1}{2}(x-1)+\frac{3}{8}(x-1)^2-\frac{15}{48}(x-1)^3+\cdots$$