Q1: A system is composed of four components, each of which is either working or failed. Consider an experiment that consists of observing the status of each component, and let the outcome of the experiment be given by the vector (x1; x2; x3; x4) where xi is equal to 1 if component i is working and is equal to 0 if component i is failed.

a) How many outcomes are in the sample space of this experiment?
b) Suppose that the system will work if components 1 and 2 are both working, or if components 3 and 4 are both working. Specify all the outcomes in the event that the system works.
c) Let E be the event that components 1 and 3 are both failed. How many outcomes are contained in event E?

Solution:

(a) $2^4 = 16$

(b) $E = \{ (1,1,0,0), (1,1,0,1), (1,1,1,0), (1,1,1,1), (1,0,1,1), (0,1,1,1), (0,0,1,1) \}$

(c) Components 2 and 4 could be working or failed, so each of these components has two possible situations

\# outcomes in $E = 2^2 = 4$

$E = \{ (0,0,0,0), (0,1,0,0), (0,0,0,1), (0,1,0,1) \}$

\###

Q2: Each of 2 balls is painted black or gold and then placed in an urn. Suppose that each ball is colored black with probability 0.5, and that these events are independent.

a) Suppose that you obtain information that the gold paint has been used (and thus at least one of the balls is painted gold). Compute the conditional probability that both balls are painted gold.
b) Suppose, now, that the urn tips over and 1 ball falls out. It is painted gold. What is the probability that both balls are gold in this case?

Solution:

B = Ball is black, G = the ball is gold

$S = \{BB, BG, GB, GG\}$, each outcome has probability $\frac{1}{4}$

$1G$ = at least one ball is gold $= \{BG, GB, GG\}$.

(a) $P(GG \mid 1G) = \frac{1}{3}$ since any coloring is equally likely

(b) FG = the balls that falls out is gold. By Bayesian formula

$$P(GG \mid FG) = \frac{P(FG \mid GG)\, P(GG)}{\cancel{P(FG)}\; P(FG \mid BB)\, P(BB) + P(FG \mid BG)\, P(BG) + P(FG \mid GB)\, P(GB) + P(FG \mid GG)\, P(GG)}$$

$$= \frac{1 \cdot \frac{1}{4}}{\frac{1}{4}\left(0 + \frac{1}{2} + \frac{1}{2} + 1\right)} = \frac{\frac{1}{4}}{\frac{1}{4} \times 2} = \frac{1}{2}.$$

##

Q3: While at Manchester United Football Club in England, Cristiano Ronaldo had about an 80% success rate of scoring penalty goals from his penalty kick attempts from the penalty spot. Assuming he kept scoring penalties at the same rate, and that all penalty kicks are independent, answer the following questions:

a) What is the probability that his first missed penalty kick will happen on his 4th penalty kick attempted?
b) What is the probability that Ronaldo would have to take more than 8 penalty kicks before missing his first attempt?
c) What is the probability that his second missed penalty kick will happen on his 8th penalty kick taken?
d) What is the expected number of penalty kicks that Ronaldo would have to take before missing his second penalty kick?

Solution:

The random variable in this question has a binomial distribution.

(a) $P = (0.2)^3 (0.8) =$

(b) $P = (0.8)^8$ (if we consider at least 8 kicks)

(c) $P = \binom{7}{1} (0.8)^6 (0.2)^2$

(d) If we define X, as the number of penalty kicks before missing his second penalty, then:

$$P(X=i) = \binom{i}{1} \times (0.2)^2 \times (0.8)^{i-1} \qquad i \geq 1$$

(otherwise 0)

$$\Rightarrow E(X) = \sum_{i=1}^{\infty} i\, P(X=i) = \sum_{i=1}^{\infty} i \binom{i}{1} (0.2)^2 (0.8)^{i-1}$$

$$= 0.04 \sum_{i=1}^{\infty} i^2 (0.8)^{i-1} = \frac{0.04}{0.8} \sum_{i=1}^{\infty} i^2 (0.8)^i$$

$$\sum_{K=1}^{\infty} K^2 z^K = \frac{z(1+z)}{(1-z)3}$$

$$\Rightarrow E(X) = \frac{0.04}{0.8} \times \frac{0.8 \times (1.8)}{(0.2)3} = 9$$

##

Q4

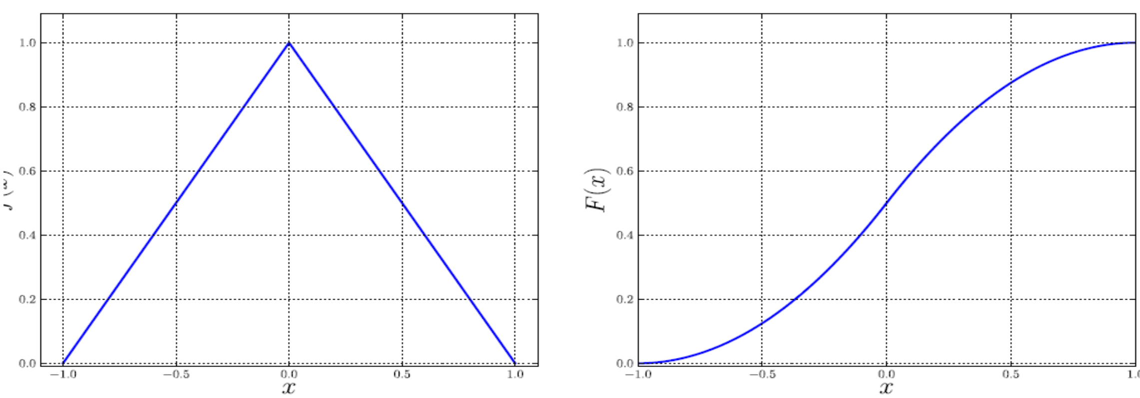
EXERCISE : For the random variable X with density function

$$f(x) = \begin{cases} x+1\ , & -1 < x \leq 0 \\ 1-x\ , & 0 < x \leq 1 \\ 0\ , & \text{otherwise} \end{cases}$$

- Draw the graph of $f(x)$
- Verify that $\int_{-\infty}^{\infty} f(x)\, dx = 1$
- Determine the distribution function $F(x)$
- Draw the graph of $F(x)$
- Determine $E[X]$
- Compute $Var(X)$ and $\sigma(X)$
- Determine $P(X \geq \frac{1}{3})$
- Determine $P(\mid X \mid \leq \frac{1}{3})$

Solution:

SOLUTION : (the graphs of $f(x)$ and $F(x)$ $\cdots$)



$$f(x) = \begin{cases} x+1 \\ \\ 1-x \end{cases}, \quad F(x) = \begin{cases} \frac{1}{2}x^2 + x + \frac{1}{2} \\ \\ -\frac{1}{2}x^2 + x + \frac{1}{2} \end{cases}, \quad \text{for} \begin{cases} -1 < x \leq 0 \\ \\ 0 < x \leq 1 \end{cases}$$

SOLUTION : (continued $\cdots$)

$$f(x) = \begin{cases} x+1 \\ \\ 1-x \end{cases} , \quad F(x) = \begin{cases} \frac{1}{2}x^2 + x + \frac{1}{2} \\ \\ -\frac{1}{2}x^2 + x + \frac{1}{2} \end{cases} , \text{ for } \begin{cases} -1 < x \leq 0 \\ \\ 0 < x \leq 1 \end{cases}$$

- $E[X] = 0$,
- $Var(X) = \frac{1}{6}$, $\quad \sigma(X) = \frac{1}{\sqrt{6}}$,
- $P(X \geq \frac{1}{3}) = F(1) - F(\frac{1}{3}) = 1 - \frac{7}{9} = \frac{2}{9}$,
- $P(\mid X \mid \leq \frac{1}{3}) = 1 - 2 \cdot \frac{2}{9} = \frac{5}{9}$.

##

Q5:

EXERCISE :

For the random variables X , Y with *joint density function*

$$f(x,y) = \begin{cases} 45xy^2(1-x)(1-y^2) , & 0 \leq x \leq 1 , \ 0 \leq y \leq 1 \\ 0 , & \text{otherwise} \end{cases}$$

- Verify that $\int_0^1 \int_0^1 f(x,y)\ dy\ dx = 1$.
- Determine the *marginal density functions* $f_X(x)$ and $f_Y(y)$.
- Are X and Y *independent* ?
- What is the value of $Cov(X,Y)$?

Solution:

SOLUTION :

- $$\int_0^1 \int_0^1 f(x,y)\ dy\ dx = \int_0^1 \int_0^1 45xy^2(1-x)(1-y^2)\ dy\ dx$$
$$= 45 \cdot \int_0^1 \Big\{ x(1-x) \int_0^1 y^2(1-y^2)\ dy \Big\}\ dx$$
$$= 45 \cdot \int_0^1 x(1-x)\ dx \ \cdot \ \int_0^1 y^2(1-y^2)\ dy$$
$$= 45 \cdot (\tfrac{x^2}{2} - \tfrac{x^3}{3})|_0^1 \ \cdot \ (\tfrac{y^3}{3} - \tfrac{y^5}{5})|_0^1$$
$$= 45 \cdot \tfrac{1}{6} \cdot \tfrac{2}{15} \ = \ 1 \ .$$

- $$f_X(x) = \int_0^1 f(x,y)\ dy = \int_0^1 45xy^2(1-x)(1-y^2)\ dy$$
$$= 45x(1-x) \ \int_0^1 y^2(1-y^2)\ dy$$
$$= 45x(1-x) \cdot \tfrac{2}{15} \ = \ 6x(1-x) \ .$$

SOLUTION : (continued $\cdots$)

- $$f_Y(y) = \int_0^1 f(x,y)\ dx = \int_0^1 45xy^2(1-x)(1-y^2)\ dx$$
$$= 45y^2(1-y^2) \ \int_0^1 x(1-x)\ dx$$
$$= 45y^2(1-y^2) \cdot \tfrac{1}{6} \ = \ \tfrac{15}{2} y^2(1-y^2) \ .$$

- X and Y are independent, since by the above
$$f(x,y) \ = \ f_X(x) \cdot f_Y(y) \ .$$

- By independence $Cov(X,Y) = 0$.

##

Q6:

EXERCISE :

The score of students taking an examination is a random variable with mean $\mu = 65$ and standard deviation $\sigma = 5$.

- What is the probability a student scores between 55 and 75 ?
- How many students would have to take the examination so that the probability that their average grade is between 60 and 70 is at least 80% ?

  HINT : Defining

  $$\bar{X} = \frac{1}{n}(X_1 + X_2 + \cdots + X_n) , \quad (\text{ the average grade })$$

  we have
  $$\mu_{\bar{X}} = E[\bar{X}] = \frac{1}{n}\, n\, \mu = \mu = 65 ,$$

  and, assuming independence,

  $$Var(\bar{X}) = n\,\frac{\sigma^2}{n^2} = \frac{\sigma^2}{n} = \frac{25}{n} , \quad \text{and} \quad \sigma_{\bar{X}} = \frac{5}{\sqrt{n}} .$$

Solution:

SOLUTION :

- By Chebyshev's inequality, with $\mu = 65$ and $\sigma = 5$,

  $$P\{\ |X - 65| \ \geq\ 10\ \} = P\{\ |X - 65| \ \geq\ 2\sigma\ \} \leq \frac{1}{2^2} = 0.25 ,$$

  so that $P\{\ |X - 65| \ <\ 10\ \} > 1 - 0.25 = 0.75.$

- With $\mu_{\bar{X}} = \mu = 65$, and (by independence) $\sigma_{\bar{X}} = 5/\sqrt{n}$, we can write

  $$P\{\ |\bar{X} - 65| \ \geq\ 5\ \} = P\{\ |\bar{X} - 65| \ \geq\ k \cdot \sigma_{\bar{X}}\ \} ,$$

  provided we let

  $$k \cdot \sigma_{\bar{X}} = 5 , \quad i.e. , \quad k = \frac{5}{\sigma_{\bar{X}}} = \sqrt{n} ,$$

  Then, by Chebyshev's inequality,

  $$P\{\ |\bar{X} - 65| \ \geq\ 5\ \} \leq \frac{1}{k^2} = \frac{1}{n} ,$$

  which is less than or equal to $1/5$ (20%) if $n \geq 5$.

###

Q7 Suppose the probability that a randomly selected person is left-handed is 0.10. In a class of 250 students just how many left-handed seats should we have to be 95% sure that no left-handed person goes without a seat?

Solution:

Since we have a binomial random variable with parameter $p = 0.1$, $n = 250$

We define random variable X as the number of left handed students:

Approximating with a Gaussian density,

$E(X) = np = 250 \times 0.1 = 25$

$var(X) = np(1-p) = 250 \times 0.1 \times 0.9 = 22.5$

$P(X < n_x) = P(X < n_x + 0.5) = 0.95$

$$P(X < n_x + 0.5) = P\left(\frac{X - 25}{\sqrt{22.5}} < \frac{n_x + 0.5 - 25}{\sqrt{22.5}}\right)$$

$$= \Phi\left(\frac{n_x + 0.5 - 25}{\sqrt{22.5}}\right) = 0.95 \Rightarrow$$

$$\frac{n_x + 0.5 - 25}{4.74} = \Phi^{-1}(0.95) = 1.65 \Rightarrow$$

$$n_x = 1.65 \times 4.74 + 25 - 0.5 = 32.32 \Rightarrow \boxed{n_x = 33}$$

##

Q8:

EXERCISE : Suppose

$$X_1 \ , \ X_2 \ , \ \cdots \ , \ X_n \ ,$$

are identical, independent, *normal* random variables, with

$$\text{mean} \quad \mu = 7 \qquad , \qquad \text{standard deviation} \quad 4 \ .$$

Let

$$\bar{X} \quad \equiv \quad \frac{1}{n} \, (X_1 \ + \ X_2 \ + \ \cdots \ + \ X_n) \ .$$

Use the CLT to determine at least how big n must be so that

- $P(| \ \bar{X} - \mu \ | \quad \leq \ 1) \quad \geq \quad 90 \ \% \ .$

Solution:

SOLUTION :

$X_1, X_2, \cdots, X_n$, are identical, independent, *normal* , on $(-\infty, \infty)$.

Each X_i has mean $\mu = 7$, and standard deviation $\sigma = 4$.

Thus $\bar{X}$ has mean $\mu_{\bar{X}} = 7$, and standard deviation $\sigma_{\bar{X}} = \frac{4}{\sqrt{n}}$.

We have

$$P(| \ \bar{X} - \mu \ | \quad \leq \ 1) \ = \ 1 \ - \ 2 \ \Phi(\frac{6-7}{4/\sqrt{n}}) \ = \ 1 \ - \ 2 \ \Phi(- \frac{\sqrt{n}}{4}) \ ,$$

so we need n so that

$$\Phi(- \frac{\sqrt{n}}{4}) \ \leq \ 0.05 \ .$$

From the Standard Normal Table we find that is sufficient to take

$$\frac{\sqrt{n}}{4} \ = \ 1.65 \ , \quad \text{which gives} \quad n \ = \ 44 \ .$$

##

Q9: In a certain chemical process, it is very important that a particular solution that is to be used as a reactant have a pH of exactly 8.20. A method for determining the pH that is available for solutions of this type is known to give measurements that are normally distributed with a mean equal to the actual pH and with a standard deviation of 0.02. Suppose 10 independent measurements yielded the following pH values:

8.18 8.17 8.16 8.15 8.17 8.21 8.22 8.16 8.19 8.18

a) What conclusion can be drawn at the 1 = 0:10 level of significance?

b) What about at the 2 = 0:05 level of significance?

(c) What is the p-value?

Solution:

Q9 :

$\sigma = 0.02$, $\mu_0 = 8.20$, $n = 10$

$\Rightarrow$ construct hypothesis

$H_0 : \mu = \mu_0$

$H_1 : \mu \neq \mu_0$

$$\bar{x} = \frac{\sum_{K=1}^{n} x_K}{n} = 8.179$$

a) $\alpha = 0.1$

$$\mu_0 \in \left[\bar{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} , \bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right]$$

H_0 is accepted if $\mu_0 \in$

$$\left[\bar{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} , \bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right]$$

Now, $Z_{\frac{0.01}{2}} = Z_{0.005}$

$\rightarrow 1 - 0.05$

$= 0.95$

Now

$$\left[8.179 - \frac{1.65 * 0.02}{\sqrt{10}}, \frac{8.179 + 1.65 * 0.02}{\sqrt{10}}\right]$$

Now, we go to table and find which Z value gives 0.95

$\Phi^{-1}(0.95) = 1.65$

$[8.179 - 0.0104, 8.179 + 0.0104]$

$[8.1686, 8.1894]$

$\rightarrow$ So we reject μ_0

b) we will be Same solution as (a)

but $\alpha = 0.05$

So $Z_{0.025} \Rightarrow \Phi^{-1}(0.975)$

$= 1.96$

c) To find P-Value

$$\text{P-Value} = 2\left(1 - \Phi(|Z_0|)\right)$$

$$Z_0 = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} = \frac{8.179 - 8.2}{0.02/\sqrt{10}}$$

##

Q10: The lifetime of a certain type of emergency batteries is normally distributed. A sample of 64 batteries has produced a sample mean of 756.4 hours and a sample standard deviation of 33.9 hours.

(a) Test the null hypothesis $H_0 : \mu \leq 747.5$ versus alternate hypothesis $H_1 : \mu > 747.5$ at the significance level of $\alpha_1 = 0.05$.

(b) Test the null hypothesis $H_0 : \mu \leq 747.5$ versus alternate hypothesis $H_1 : \mu > 747.5$ at the significance level of $\alpha_1 = 0.01$.

(c) Find the p-value of the test, and use the p-value to verify your answers.

Solution:

Q10:

σ is unknow so T-distrb but $n = 64$ So we use normal dist

$n = 64$, $\bar{x} = 756.4$, $S = 33.9$

$H_0: \mu \leq 747.5$

$H_1: \mu > 747.5$

$\alpha_1 = 0.05$

P-value $< \alpha$ H_0 should be rejected

P-Value $= \Phi(z_0)$

$$z_0 = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$$

but here we use S instead of σ

$\Downarrow$

$$z_0 = \frac{\bar{x} - \mu_0}{S/\sqrt{n}}$$

###

Q11: All pacemakers generate a pulse of current to depolarize a bit of the heart, the wave then spontaneously spreads to the rest of that part of the heart. Pulse duration is programmable in all currently manufactured permanent pacemakers. The shorter the pulse, the greater the necessary current. The length of the pulse for a particular setting will vary due to random fluctuations such that the actual pulse length is normally distributed with a mean equal to the setting and with an unknown variance. A type of pacemaker is claimed to be reliable since its standard deviation standard deviation is less than 0.10. If a random sample of 20 pacemakers of that type resulted in a sample standard deviation of 0.08, should this claim be accepted? Suppose the level of significance is alpha = 0:05.

a) State the appropriate choice of the null hypothesis, and the alternate hypothesis.
b) Compute the value of the test statistic, X^2_{19}
c) Determine its p-value.
d) Based on this evidence, should the claim be accepted? Also, comment on the appropriate choice of a significance level for this problem.

Solution:

Q11 :

Here the hypothesis is about σ which means, we don't know μ.
It is chi-squared

a) $H_0 : \sigma \leq 0.10$
$H_1 : \sigma > 0.10$

b) $\alpha = 0.05$, $n = 20$

$\chi^2_{19} \Longrightarrow$ if P-Value $< \alpha$ rejected

to find P-Value

$$\text{P-Value} = 1 - \underbrace{F(\chi^2_{19})}_{\Downarrow}$$

then we go to χ^2 table

###

Q12: Give a method for generating random numbers with a distribution function

$$F(x) = \frac{1}{2}(x + x^2), \quad 0 < x < 1.$$

###

Q13: Give a method for generating a random variable having density function

$$f(x) = \frac{e^x}{e - 1}, 0 < x < 1$$

Using 10000 simulated values from this distribution, estimate its mean and variance.

Solution:

Using 10000 simulated values from this distribution, estimate its mean and variance. Use inverse transform. In this case $F(x) = (e^x - 1)/(e - 1)$ and $F^{-1}(U) = \ln(1+U(e-1))$. x=log(1+rand(1,10000)*(exp(1)-1)); mean(x)=0.58 and var(x)=.0792. The true values are $\int_0^1 x\frac{e^x}{e-1}dx = 0.58198$ and the variance is $\int_0^1 x^2\frac{e^x}{e-1}dx - (0.58198)^2 := 0.07932\,3$.

###

Q14: Give a method for generating a random variable having density function

$$f(x) = \begin{cases} \frac{x-2}{2} & \text{if} \quad 2 \leq x \leq 3 \\ 1 - \frac{x}{6} & \text{if} \quad 3 \leq x \leq 6 \end{cases}$$

Using 10000 simulated values from this distribution, estimate its mean and variance. Apply the inverse transform method.

Solution:

Using 10000 simulated values from this distribution, estimate its mean and variance. Apply the inverse transform method. In this case

$$F(x) = \begin{cases} \frac{(x-2)^2}{4} \text{ if } 2 \le x \le 3 \\ \frac{1}{4} + (x-3)(\frac{3}{4} - \frac{x}{12}) \text{ if } 3 < x < 6 \end{cases}$$

Inverting this c.d.f. gives

$$F^{-1}(U) = \begin{cases} 2 + 2\sqrt{U} \text{ if } 0 \le U \le 1/4 \\ 6 - 2\sqrt{3 - 3U} \text{ if } U > 1/4 \end{cases}$$

##