Midterm MATH 251, October 17, 2013

Justify all answers

Problem 1 [5 pt] Are the following subsets of $\mathcal{P}(\mathbb{R})$ (space of polynomials) subspaces? **Justify your answer**.

$$(\mathbf{a})\ \ W_1 := \{p(x) :\ \ p(4) = 0,\ p(3) = 0\ \} \qquad (\mathbf{b})\ \ W_2 := \{p(x) :\ \ p(2) = 1\} \tag{1}$$

Problem 2 [5 pt] Find a basis and compute the dimension for the following subspace of $\mathbb{R}^5$

$$W := \{\langle a_1, a_2, a_3, a_4, a_5\rangle :\quad a_1 + a_2 + a_4 + a_5 = 0\ ,\quad a_3 + a_4 + a_5 = 0\} \tag{2}$$

Problem 3 [5 pt] Let $V = Mat_{2\times 3}(\mathbb{R})$ and consider the following subspaces:

$$W_1 := \left\{ \begin{bmatrix} a & a-b & a+b \\ c+d & d & d \end{bmatrix} \ \ a,b,c,d \in \mathbb{R} \right\}\ , \qquad W_2 := \left\{ \begin{bmatrix} f & -f & g \\ e & e & \ell - e \end{bmatrix} \ \ e,f,g,\ell \in \mathbb{R} \right\} \tag{3}$$

Find bases and dimensions of $W_1,\ W_2,\ W_1 + W_2,\ W_1 \cap W_2$.

Problem 4 [5 pt] Let $T : V \to W$ be a linear transformation. Suppose that it is one-to-one. Let $\{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k\}$ be a linearly independent subset of V. Prove that $\underline{w}_1 = T\underline{v}_1, \ldots, \underline{w}_k = T\underline{v}_k$ is a linearly independent subset of W. Is the assumption of T being one-to-one necessary for the conclusion that the $\underline{w}_1, \ldots, \underline{w}_k$ are independent? Explain.

Problem 5 [5 pt] Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by $T(\langle a_1, a_2\rangle) = \langle 2a_1 - a_2, a_1 + a_2, 5a_1 - a_2\rangle$. Let β be the standard basis of $\mathbb{R}^2$ and $\alpha = (\langle 1, 1\rangle, \langle 1, -1\rangle)$ another basis of $\mathbb{R}^2$. Let $\gamma = (\langle 1, 2, 3\rangle, \langle 0, 1, 2\rangle, \langle 0, 0, 1\rangle)$ be a basis of $\mathbb{R}^3$. Compute $[T]_\beta^\gamma,\quad [T]_\alpha^\gamma$.

Problem 6 [5 pt]

Consider the transformation $T : \mathcal{P}_3 \to \mathcal{P}_6$ where $\mathcal{P}_n$ denotes the finite dimensional vector space consisting of polynomials of degree up to n:

$$T(p(x)) = p(x^2) + p(x-1) \tag{4}$$

Note: here $p(x-1)$ means the shift of variable, for example if $p(x) = x^2 + 2$ then $p(x-1) = (x-1)^2 + 2 = x^2 - 2x + 3$; similarly $p(x^2)$ means the change of variable, for example if $p(x) = 2x^3 + x$ then $p(x^2) = 2(x^2)^3 + (x^2) = 2x^6 + x^2$.

1. Show that T is linear;
2. Find $[T]_\beta^\gamma$ where $\beta = (1, x, x^2, x^3), \gamma = (1, x, x^2, x^3, x^4, x^5, x^6)$ are the standard ordered bases of $\mathcal{P}_3, \mathcal{P}_6$.

Problem 7 [Bonus 3 pt] Let $T : V \to W$ and $U : W \to Z$ be two linear transformations between the indicated vector spaces V, W, Z. Prove that $\mathbf{R}(UT) \subseteq \mathbf{R}(U)$, where $\mathbf{R}(U),\ \mathbf{R}(UT)$ denote the *ranges* of the indicated transformations. Give an example where the inclusion is strict.