TOPIC: WAITING LINE ANALYSIS

Q 1: Fore and Aft Marina is a new marina planned for a location on the Ohio River near Madison, Indiana. Fore and Aft expects a mean arrival rate of 7 boats per hour. Fore and Aft is evaluating the following two options:

- Option A: One dock with a mean service rate of 12 boats per hour.
- Option B: Two docks with a single queue. Each dock is capable of serving on average 10 boats per hour.

For both the options (Option A and Option B):

a) What is the average time a boat will wait for service?
b) What is the probability that more than 2 boats are waiting for services?
c) What is the probability that a boat has to wait before getting the services?

Solution:

Option A:

- Arrival rate (λ) = 7 boats/hour
- Service rate (μ) = 12 boats/hour
- No. of ~~Servers~~ = 1
- Hence, the problem can be solved using the single server model with exponential service durations and infinite number in the system.

➢ Average time a boat will wait for service, $W_q = \frac{L_q}{\lambda} = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{7}{12(12-7)} = 0.1166\, hrs = 7\, min$

➢ Probability that more than 2 boats are waiting for services implies that there are 3 boats in the system. Hence, this can be calculated as follows:

Probability of more than 3 boats in the system

$= P_4 + P_5 + P_6 \ldots = 1 - (P_0 + P_1 + P_2 + P_3)$) =1 - 0.8829 = 0.1171

- Probability of no boats in the system $P_0 = 1 - \frac{\lambda}{\mu} = 1 - \frac{7}{12} = 0.4166$
- Probability of 1 boat waiting for service $P_1 = \frac{\lambda}{\mu}\left(1 - \frac{\lambda}{\mu}\right) = \frac{7}{12}\left(1 - \frac{7}{12}\right) = 0.243$
- Probability of 2 boats in the system $P_2 = \left(\frac{\lambda}{\mu}\right)^2 \left(1 - \frac{\lambda}{\mu}\right) = 0.141$
- Probability of 3 boats in the system $P_3 = \left(\frac{\lambda}{\mu}\right)^3 \left(1 - \frac{\lambda}{\mu}\right) = 0.08225$

➢ Probability that a boat has to wait before getting the services is same as the probability that the server is busy. This is also the probability that there is 1 boat and more than 1 boat in the system.

- Probability of the server is busy = ρ = 1 - $P_0 = \frac{\lambda}{\mu} = \frac{7}{12} = 0.5833$
- Probability of 1 boat or more in the system $= P_1 + P_2 + P_3 + \cdots = 1 - P_0 = \frac{\lambda}{\mu} =$ 0.5833

Option B:

- Arrival rate (λ) = 7 boats/hour
- Service rate (μ) = 10 boats/hour/dock
- No. of Servers = 2
- Hence, the problem can be solved using the multiple-server model.
- $P_0 = \frac{1}{\left[\sum_{n=0}^{n=M-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\right] + \frac{1}{M!}\left(\frac{\lambda}{\mu}\right)^M \frac{M\mu}{M\mu-\lambda}} = \frac{1}{\left[1+\frac{7}{10}\right] + \frac{1}{2!}\left(\frac{7}{10}\right)^2 \frac{2*10}{2*10-7}} = 0.481$ (This can also be found from the P_0 table with M=2)

- The expected queue length and the average time before service can be computed as:

$L_q = \frac{\lambda\mu\left(\frac{\lambda}{\mu}\right)^M}{(M-1)!(M\mu-\lambda)^2} P_o = \frac{7*10*(0.7)^2}{(20-7)^2} \times 0.481 = 0.10$ boats

$W_q = \frac{L_q}{\lambda} = \frac{0.10}{7}$ = 0.01428 hours = 0.857 minutes

- Probability that more than 2 boats are waiting for services implies that there are 4 boats in the system (2 in service and 2 waiting in line). Hence, this can be calculated as follows:

$$P_n = \begin{cases} \frac{1}{M!\,M^{n-M}}\left(\frac{\lambda}{\mu}\right)^n P_0 \text{ for } n > M \\ \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n P_0 \text{ for } n \leq M \end{cases}$$

- $P_1 = \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n P_0 = 0.7 * 0.481 = 0.337$
- $P_2 = \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n P_0 = 0.5*(0.7)^2 * 0.481 = 0.118$
- $P_3 = \frac{1}{M!M^{n-M}}\left(\frac{\lambda}{\mu}\right)^n P_0 = \frac{1}{2*2}(0.7)^3 0.481 = 0.0412$
- $P_4 = \frac{1}{M!M^{n-M}}\left(\frac{\lambda}{\mu}\right)^n P_0 = \frac{1}{2*2^2}(0.7)^4 0.481 = 0.0144$
- Probability that more than 2 boats are waiting for services =
- Probability of more than 4 boats in the system
  $= P_5 + P_6 + \cdots = 1 - (P_0 + P_1 + P_2 + P_3 + P_4)$ =1 - (0.481+0.337+0.118+0.0412+0.0144) =0.0084

- Probability that a boat has to wait before getting the services = $\frac{W_q}{W_a}$

  $W_a = \frac{1}{M\mu-\lambda} = \frac{1}{2*10-7} = 0.0769$

  $P_w = \frac{W_q}{W_a} = \frac{0.01428}{0.0769} = 0.1857$

Q 2: A warehouse has three loading docks. Trucks wait in a single line until signaled to enter in the next available dock. Currently every dock requires a team of two persons who can load a truck on average in 10 minutes. Trucks arrive at the rate of 12 per hour.

a) Determine the expected queue length as well as the average time before a truck is assigned to a dock.
b) Determine the probability that a truck get the dock without waiting in the queue.
c) It is estimated that each truck waiting in the system costs $30/hour. The warehouse manager is evaluating the following two options listed below. Which option would you recommend i.e. current system, option 1 or option 2? Justify your answer by actual calculations. (Assume warehouse operates 10 hours per day and 300 days per year)
   - Option 1: One Automated Loading System (fixed service time), Loading Rate: 20 trucks per hour, Total Cost = $35000/year
   - Option 2: One Semi-automated loading System, Loading Rate: 15 trucks per hour, Total Cost = $18000/year

Solution:

- Arrival rate (λ) = 12 trucks/hour
- Service rate (μ) = 60/10 = 6 trucks/hour/dock
- No. of Servers = M = 3

- Because all the trucks wait in a single line and there are three docks, the problem can be solved using the multiple-server model.

- $P_0 = \frac{1}{\left[\sum_{n=0}^{n=M-1}\frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\right]+\frac{1}{M!}\left(\frac{\lambda}{\mu}\right)^M\frac{M\mu}{M\mu-\lambda}} = \frac{1}{\left[1+2+\frac{1}{2}(2)^2\right]+\frac{1}{3!}(2)^3\frac{3*6}{3*6-12}} = \frac{1}{4+5} = \frac{1}{9} = 0.1111$ (This can also be found from the P_0 table with M=3)

(a) The expected queue length and the average time before a truck is assigned to a dock can be computed as:

- $L_q = \frac{\lambda\mu\left(\frac{\lambda}{\mu}\right)^M}{(M-1)!(M\mu-\lambda)^2}P_o = \frac{12*6(2)^3}{2(18-12)^2} \times 0.1111 = 0.89$ trucks
- $W_q = \frac{L_q}{\lambda} = \frac{0.89}{12} = 0.0741$ hours = 4.45 minutes

(b) The probability that a truck arriving at the dock must wait for service is $P_w = \frac{W_q}{W_a}$

- $W_a = \frac{1}{M\mu-\lambda} = \frac{1}{3*6-12} = 0.1667$
- $P_w == \frac{W_q}{W_a} = \frac{0.0741}{0.1667} = 0.4445$

Therefore, the probability that a truck gets the dock without waiting in the queue $= 1 - P_w = 0.5555$

(c) Note that we are given the truck's waiting cost in the system. Hence, we will compute the average waiting time in the system for the three options to compare them.

- $W_{s-CURRENT\ SYSTEM} = W_q + \frac{1}{\mu} = 0.0741 + 0.1666 = \mathbf{0.240\ hrs} = 14.45\ min$

Option 1: This option of one automated loading system with fixed service time can be modeled appropriately using single-server model with constant service time.

- Arrival rate (λ) = 12 trucks/hour
- Service rate (μ) = 20 trucks/hour
- $W_q = \frac{L_q}{\lambda} = \frac{\lambda}{2\mu(\mu-\lambda)} = \frac{12}{2*20(20-12)} = 0.0375\ hrs = 2.25\ min$
- $W_{s-OPTION\ 1} = W_q + \frac{1}{\mu} = 0.0375 + 0.05 = \mathbf{0.0875\ hrs} = 5.25\ min$

Option 2: This option of one semi-automated loading system can be modeled appropriately using single-server model with exponential service durations and infinite system capacity.

- Arrival rate (λ) = 12 trucks/hour
- Service rate (μ) = 15 trucks/hour
- $W_q = \frac{L_q}{\lambda} = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{12}{15(15-12)} = 0.2666\ hrs = 16\ min$
- $W_{s-OPTION\ 2} = W_q + \frac{1}{\mu} = 0.2666 + 0.0666 = \mathbf{0.333\ hrs} = 20\ min$

	Current System	**Option 1**	**Option 2**
Arrival Rate	12 trucks/hour	12 trucks/hour	12 trucks/hour
Expected No. of Truck arrivals/year	12*10*300 = 36,000	12*10*300 = 36,000	12*10*300 = 36,000
Average Waiting Time in System	0. 240 hours	0.0875 hours	0.333 hours

Waiting Cost	$30/hour	$30/hour	$30/hour
Total Waiting Cost/year	36,000*0.240*30 = $ 259,200	36,000*0.0875*30 = $ 94,500	36,000*0.333*30 = $ 359,640
Annual Fixed Cost	-	$ 35,000	$ 18,000
TOTAL ANNUAL COST	**$ 259,200**	**$ 129,500 (LOWEST)**	**$ 377,640**

Q 3: The Chattanooga Furniture store gets an average of 50 customers per shift. Marilyn Helms, the manager, wants to calculate whether she should hire 1, 2, 3 or 4 salespeople. She has determined that average waiting times will be 7 minutes with one salesperson, 4 minutes with two salespeople, 3 minutes with three salespeople, and 2 minutes with four salespeople. She has estimated the cost per minute that customer wait at $1. The cost per salesperson per shift (including fringe benefits) is $70. How many sales people should be hired?

Solution:

- Arrival rate (λ) = 50 customer/shift
- Cost of Waiting = $1/min/customer
- Cost per Salesperson per Shift = $70
- The costs associated with four options are as follows:

	Option 1: 1 Salesperson	**Option 2: 2 Salespersons**	**Option 3: 3 Salespersons**	**Option 4: 4 Salespersons**
Average Waiting Time	7 min	4 min	3 min	2 min
Total Cost of Waiting	50*1*7 = $ 350	50*1*4 = $ 200	50*1*3 = $ 150	50*1*2 = $ 100
Labor Cost	70*1 = $ 70	70*2 = $ 140	70*3 = $ 210	70*4 = $ 280
TOTAL COST	$420	$340 (LOWEST)	$360	$380

- The optimal policy is to hire 2 salespersons as it has the lowest total cost.

Q 4: A hockey arena experiences the highest volume of ticket sales on Fridays from 10:00 a.m. to 6:00 p.m. Currently, there are three ticket windows serving their own line in front of each window. A consultant hired recently carried out a time study and found that the verbal communication between the customer and the salesperson takes three minutes on the average. The same study also revealed that there were no customers in the system 20% of the time. Based on Poisson arrival rates and negative exponential service times, answer the following questions.

a) What is the arrival rate per hour at each line?
b) How long on the average must a customer wait before the start of the service by the salesperson?
c) What is the average length of each queue?
d) What is the probability of finding at most two customers waiting in each queue?

After studying the sales operations, the same consultant proposed having a single queue to be served by three sales persons.

e) What is the arrival rate for this configuration?
f) What is the probability of finding no customers in the system?
g) How long on the average must a customer wait before the start of service by the salesperson?
h) Calculate the total amount of customer waiting time saved on Fridays if the consultant's proposal is accepted.

Solution:

- Service rate (μ) = 20 customers/hour
- Probability that there were no customers in the system = $P_0 = 0.20$
- Note that there are three ticket windows serving their own line in front of each window. This implies that this is a single-server model. Furthermore, all the three systems are identical; hence it suffices to analyze one of the three systems.

For a single server model, $P_0 = 1 - \frac{\lambda}{\mu} \rightarrow 0.20 = 1 - \frac{\lambda}{20}\ or\ \lambda = 16\ customers/hour$

- Hence, arrival rate at each line is 16 customers per hour.
- $W_q = \frac{L_q}{\lambda} = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{16}{20(20-16)} = 0.20\ hrs = 12\ min$
- $L_q = \lambda W_q = 16 * 0.20 = 3.2\ customers$
- Probability of finding at most 2 customers <u>waiting in queue</u> is same as the probability of finding at most 3 customers <u>in the system</u>. Hence, this can be calculated as follows:

$$= P_{\leq 3} = P_0 + P_1 + P_2 + P_3$$
$$= (1-0.8) + 0.8 * (1-0.8) + 0.8^2 * (1-0.8) + 0.8^3 * (1-0.8) = 0.5904$$

If the consultant proposes having a single queue to be served by three sales persons, then this becomes a multiple-server model with

- Arrival rate (λ) = 16+16+16 = 48 customers/hour
- Service rate (μ) = 20 trucks/hour/salesperson
- No. of Servers (M) = 3

Probability of finding no customers in the system is

- $P_0 = \frac{1}{\left[\sum_{n=0}^{n=M-1}\frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\right] + \frac{1}{M!}\left(\frac{\lambda}{\mu}\right)^M \frac{M\mu}{M\mu-\lambda}} = \frac{1}{\left[1+2.4+\frac{1}{2}(2.4)^2\right]+\frac{1}{3!}(2.4)^3\frac{3*20}{3*20-48}} = \frac{1}{6.28+11.51} = 0.056$

The average time a customer must wait before the start of service by the salesperson can be computed as:

- $L_q = \frac{\lambda\mu\left(\frac{\lambda}{\mu}\right)^M}{(M-1)!(M\mu-\lambda)^2} P_o = \frac{48*20(2.4)^3}{2(60-48)^2} \times 0.056 = 2.59$ customers
- $W_q = \frac{L_q}{\lambda} = \frac{2.59}{48}$ = 0.0536 hours = 3.24 minutes

Note that a customer waits on an average for 12 minutes in queue in the current system, whereas the average waiting time reduced to 3.24 minutes in the proposed system. Hence, there is a reduction of 8.86 minutes or 0.1477 hours of waiting time per customer. This amounts to 48 customers/hour*0.1477 hours/customer* 8 hours/day = 56.72 hours/day.

Q 5: Customers arrive at the lobby of the exclusive and expensive Ritz Hotel at the rate of 40 per hour (Poisson distributed) to check in. The hotel normally has three clerks available at the desk to check guests in. The average time for a clerk to check in a guest is four minutes (exponentially distributed). Clerks are paid $12 per hour and the hotel assigns a goodwill cost of $2 per minute for the time a guest must wait in the line.

a) Determine the current waiting time for guests before getting the services.
b) Determine if the present check-in system is cost effective; if it is not, recommend what hotel management should do.

<u>Solution:</u>

- Arrival rate (λ) = 40 customers/hour
- Service rate (μ) = 60/4 min per customer= 15 customers/hour/clerk
- No. of Servers = M = 3
- The problem can be solved using the multiple-server model.

Probability of finding no customers in the system is

- $P_0 = \frac{1}{\left[\sum_{n=0}^{n=M-1}\frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\right] + \frac{1}{M!}\left(\frac{\lambda}{\mu}\right)^M \frac{M\mu}{M\mu-\lambda}} = \frac{1}{\left[1+2.67+\frac{1}{2}(2.67)^2\right]+\frac{1}{3!}(2.67)^3\frac{3*15}{3*15-40}} = \frac{1}{7.23+28.55} = 0.028$

The average time a customer must wait before the start of service by the clerk can be computed as:

- $L_q = \frac{\lambda\mu\left(\frac{\lambda}{\mu}\right)^M}{(M-1)!(M\mu-\lambda)^2} P_o = \frac{40*15(2.667)^3}{2(40-15)^2} \times 0.028 = 6.38$ customers
- $W_q = \frac{L_q}{\lambda} = \frac{6.38}{40}$ = 0.1595 hours = 9.57 minutes

Clerks are paid $12 per hour and there are 3 clerks, hence the total labor cost is $36 per hour.

The arrival rate is 40 customers per hour and on an average every customer waits 9.57 minutes. The hotel assigns a goodwill cost of $2 per minute for the time each guest must wait in the line. Hence, the total waiting cost/hour = 40*9.57*2 = $765.6. This is very high compared to the labor cost of $36 per hour. Hence, the management should increase the number of clerks in order to reduce the total waiting time and goodwill costs (thereby increasing service level).

Q 6: Customers arrive at the drive-up window of a fast food restaurant at the rate of 25 per hour. The employee working at the window can serve one customer every 2 minutes. Assume Poisson arrivals and exponential service rates.

(a) What is the average utilization of the employee?
(b) What is the average number of customers in line?
(c) What is the average number of customers in the system?
(d) What is the average time spent waiting in line?
(e) What is the average time waiting in the system?
(f) What is the probability that exactly 2 cars will be waiting in line?
(g) What is the probability that an arriving customer will have an actual waiting time (in the system) of more than 20 minutes?

If the manager decides to train the employees to reduce the service time variability as much as possible, what would be the change in the above performance measures? Use appropriate model to do the calculations.

Solution:

- Arrival rate (λ) = 25 customer/hour
- Service rate (μ) = 1 customer every 2 min = 30 customers/hour.
- No. of Servers = 1
- Hence, the problem can be solved using the single server model (Model 1).

a) The average utilization of the employee = $\rho = \frac{\lambda}{\mu} = \frac{25}{30} = 0.833$ or 83.33%

b) The average number of customers in line:

$$L_q = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{25*25}{30*(30-20)} = 4.17 \text{ customers}$$

c) The average number of customers in the system:

$$L_s = L_q + \frac{\lambda}{\mu} = 4.17 + 0.83 = 5 \text{ customers}$$

d) The average time spent waiting in line = $W_q = \frac{L_q}{\lambda} = \frac{4.17}{25} = 0.1666 \text{ hours} = 10 \text{ min}$

e) The average time spent waiting in the system = $W_s = \frac{L_s}{\lambda} = \frac{5}{25} = 0.2 \text{ hours} = 12 \text{ min}$

f) Probability that exactly 2 cars will be waiting in line = Probability that there are exactly 3 cars in the system = $P_3 = \left(\frac{\lambda}{\mu}\right)^n \left(1-\frac{\lambda}{\mu}\right) = \left(\frac{25}{30}\right)^3 \left(1-\frac{25}{30}\right) = 0.096$

g) Probability that an arriving customer will have an actual waiting time (in the system) of more than 20 minutes = Probability that an arriving customer will have a waiting time in line of more than 18 minutes (0.3 hour):

$$P_{w\ in\ Q>t} = \frac{\lambda}{\mu} e^{-(\mu-\lambda)t} = \frac{25}{30} e^{-(30-25)(0.3)} = 0.1859$$

If the manager decides to train the employees to reduce the service time variability as much as possible, what would be the change in the above performance measures? Use appropriate model to do the calculations.

Reducing the service time variability as much as possible implies making the service time constant. For the single server model with constant service times, the calculations are as follows:

a) The average utilization of the employee = $\rho = \frac{\lambda}{\mu} = \frac{25}{30} = 0.833$ or 83.33% (Unchanged)

b) The average number of customers in line:

$$L_q = \frac{\lambda^2}{2\mu(\mu-\lambda)} = \frac{25*25}{2*30*(30-20)} = 2.08 \text{ customers}$$ (Reduces by Half)

c) The average number of customers in the system:

$$L_s = L_q + \frac{\lambda}{\mu} = 2.08 + 0.83 = 2.8833 \text{ customers}$$

d) The average time spent waiting in line = $W_q = \frac{L_q}{\lambda} = \frac{2.08}{25} = 0.0832 \text{ hours} = 5 \text{ min}$ (Reduces by Half)

e) The average time spent waiting in the system:

$$W_s = \frac{L_s}{\lambda} = \frac{2.8833}{25} = 0.115 \text{ hours} = 6.92 \text{ min}$$

f) Probability that exactly 2 cars will be waiting in line = Probability that there are exactly 3 cars in the system = $P_3 = \left(\frac{\lambda}{\mu}\right)^n \left(1-\frac{\lambda}{\mu}\right) = \left(\frac{25}{30}\right)^3 \left(1-\frac{25}{30}\right) = 0.096$ (Unchanged)

g) Probability that an arriving customer will have an actual waiting time (in the system) of more than 20 minutes = Probability that an arriving customer will have a waiting time in line of more than 18 minutes (0.3 hour):

$$P_{w\ in\ Q>t} = \frac{\lambda}{\mu} e^{-(\mu-\lambda)t} = \frac{25}{30} e^{-(30-25)(0.3)} = 0.1859$$ (Unchanged)

Q 7: The manager of an amusement park wants to hire a repair team for the bumper car section. On average 2 cars breakdown every hour. The loss of revenue due to breakdown is $40/hour. The average repair (service) time per car is 30 minutes for a team of size 1, 20 minutes for a team of size 2, and 15 minutes for a team of size 3.

(a) Assuming that each repair person costs $20/hour, how many people should be hired?

(b) Suppose that the director of the amusement park has asked the manager to go with the repair team size of 2. However, to keep the cars in working condition, more than one repair team may be needed. How many teams should be hired?

Solution:

- Arrival rate (λ) = 2 cars/hour
- No. of Servers = M = 1
- Note that a repair team works together to get the repair job done, hence a team acts as a server here (e.g. a team of three repair person is one single server, and not 3 servers).
- The problem can be solved using the single-server model (Model 1).

For a team of 1 repair person:

- Service time = 30 minutes/car
- Service rate (μ) = 2 cars/hour
- We cannot proceed with the calculations as the arrival rate = service rate, L_q = infinity

For a team of 2 repair persons:

- Service time = 20 minutes/car
- Service rate (μ) = 3 cars/hour
- We can proceed with the calculations as the arrival rate is less than service rate.
- The average number of cars in the system = $L_s = \frac{\lambda}{(\mu-\lambda)} = \frac{2}{(3-2)} = 2\ cars$
- Total cost per hour = 2 repair-person *$20/hour + 2 cars * $40/hour = **$120/hour**

For a team of 3 repair persons:

- Service time = 15 minutes/car
- Service rate (μ) = 4 cars/hour
- We can proceed with the calculations as the arrival rate is less than service rate.
- The average number of cars in the system = $L_s = \frac{\lambda}{(\mu-\lambda)} = \frac{2}{(4-2)} = 1\ car$
- Total cost per hour = 3 repair-person *$20/hour + 1 car * 40/hour = **$100/hour**

Hire 3 repair persons as it is least among the two feasible options that we have.

(b) Suppose that the director of the amusement park has asked the manager to go with the repair team size of 2. However, to keep the cars in working condition, more than one repair team may be needed. How many teams should be hired?

This is a multiple-sever model, because we are talking about more than one team, where each team comprises of 2 repair persons.

For a team of 2 repair persons:

- Arrival rate (λ) = 2 cars/hour
- Service rate (μ) = 3 cars/hour
- No. of Servers = M = 1
- $L_s = \frac{\lambda}{(\mu-\lambda)} = \frac{2}{(3-2)} = 2\ cars$
- Total cost per hour = 2 repair-person *$20/hour + 2 cars * 40/hour = **$120/hour**

For 2 teams of 2 repair persons each:

- Arrival rate (λ) = 2 cars/hour
- Service rate (μ) = 3 cars/hour/team
- No. of Servers = M = 2

- $P_0 = \frac{1}{\left[\sum_{n=0}^{n=M-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\right] + \frac{1}{M!}\left(\frac{\lambda}{\mu}\right)^M \frac{M\mu}{M\mu-\lambda}}$ = 0.5
- $L_s = \frac{\lambda\mu\left(\frac{\lambda}{\mu}\right)^M}{(M-1)!(M\mu-\lambda)^2} P_o + \frac{\lambda}{\mu}$ = 0.75 cars
- Total cost per hour = 2 teams * 2 repair-person *$20/hour + 0.75 car * 40/hour = **$110/hour**

For 3 teams of 2 repair persons:

- Arrival rate (λ) = 2 cars/hour
- Service rate (μ) = 3 cars/hour
- No. of Servers = M = 3
- $P_0 = \frac{1}{\left[\sum_{n=0}^{n=M-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\right] + \frac{1}{M!}\left(\frac{\lambda}{\mu}\right)^M \frac{M\mu}{M\mu-\lambda}}$ = 0.512
- $L = \frac{\lambda\mu\left(\frac{\lambda}{\mu}\right)^M}{(M-1)!(M\mu-\lambda)^2} P_o + \frac{\lambda}{\mu}$ = 0.68 cars
- Total cost per hour = 3 teams * 2 repair-person *$20/hour + 0.68 car * 40/hour = **$147.2/hour**

Note that the total cost has started increasing. Hence, having two teams of 2 repair person is optimal.

Q 8: The manager of a regional warehouse must decide on the number of loading docks to request for a new facility in order to minimize the sum of dock costs and driver-truck costs. The manager has learned that each driver-truck combination represents a cost of $300 per day and that each dock plus loading crew represents a cost of $1,100 per day.

a) How many docks should be requested if trucks arrive at the rate of four per day, each dock can handle five trucks per day assuming both rates are Poisson?

For 1 dock:

- Arrival rate (λ) = 4 trucks/day
- Service rate (μ) = 5 trucks/day
- No. of Servers = M = 1 (Single server model)
- Average no. of trucks in system = $L_s = \frac{\lambda}{(\mu-\lambda)} = \frac{4}{(5-4)} = 4\ trucks$
- Cost of 1 dock (with loading crew) per day = $1,100.
- Cost of trucks waiting (along with driver) per day = 4*300 = $1,200
- Total cost per day = **$2,300.**

For 2 docks:

- Arrival rate (λ) = 4 trucks/day
- Service rate (μ) = 5 trucks/day/dock
- No. of Servers = M = 2 (Multiple server model)
- $P_0 = \frac{1}{\left[\sum_{n=0}^{n=M-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\right] + \frac{1}{M!}\left(\frac{\lambda}{\mu}\right)^M \frac{M\mu}{M\mu-\lambda}}$ = 0.429
- Average no. of trucks in the system $= L_s = \frac{\lambda\mu\left(\frac{\lambda}{\mu}\right)^M}{(M-1)!(M\mu-\lambda)^2} P_o + \frac{\lambda}{\mu}$ = 0.95 trucks
- Cost of 2 docks (with loading crew) per day = 2*$1,100 = $ 2,200.

- Cost of trucks waiting (along with driver) per day = 0.95*300 = $285
- Total cost per day = **$2,485.**

⇨ **Hence, the optimal policy is to use one dock as it is cheaper.**

b) An employee has proposed adding new equipment that would speed up the loading rate to 5.71 trucks per day. The equipment would cost $100 per day for each dock. Should the manager invest in the new equipment?

This new equipment will change the service rate (μ) to 5.71 while the arrival rate (λ) remains at 4.

For 1 dock:

- Arrival rate (λ) = 4 trucks/day
- Service rate (μ) = 5.71 trucks/day
- No. of Servers = M = 1 (single server model)
- Average no. of trucks in the system = $L_s = \frac{\lambda}{(\mu-\lambda)} = \frac{4}{(5.71-4)} = 2.34\ trucks$
- Cost of 1 dock (with loading crew) per day = $1,100
- Cost of trucks waiting (along with driver) per day = 2.34*300 = $702.
- Cost of new equipment per day = $100.
- Total cost per day = **$1,902.**

For 2 docks:

- Arrival rate (λ) = 4 trucks/day
- Service rate (μ) = 5.71 trucks/day/dock
- No. of Servers = M = 2
- $P_0 = \frac{1}{\left[\sum_{n=0}^{n=M-1}\frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\right]+\frac{1}{M!}\left(\frac{\lambda}{\mu}\right)^M\frac{M\mu}{M\mu-\lambda}} = 0.481$
- Average no. of trucks in system $= L_s = \frac{\lambda\mu\left(\frac{\lambda}{\mu}\right)^M}{(M-1)!(M\mu-\lambda)^2}P_o + \frac{\lambda}{\mu}$ = 0.80 trucks
- Cost of 2 docks (with loading crew) per day = 2*$1,100 = $ 2,200.
- Cost of trucks waiting (along with driver) per day = 0.80*300 = $240
- Cost of two of these new equipment per day = 2*$100 = $200.
- Total cost per day = **$2,640.**

Note that the total cost has started increasing, hence having 1 dock and investing in this new equipment is optimal (compare $1,902 to $2,300).