Exercise Problems for the Midterm

MATH 265 , Winter 2013

Problem 1: Evaluate the integral

$$\iint_\Omega e^{-y^2/2} dA \ ,$$

where Ω is the region bounded by y-axis and the lines $2y = x$, $y = 1$.

Problem 2: Find the volume of the solid bounded above by the plane $z = 2x + 2$ and below by the disk $(x-1)^2 + y^2 \le 1$ in the plane $z = 0$.

Problem 3: Find the volume of the solid bounded above by the cone $z = 2 - \sqrt{x^2 + y^2}$ and below by the disk $(x-1)^2 + y^2 \le 1$. Hint: use polar coordinates.

Problem 4: Evaluate the integral

$$\int_0^{1/2} \int_0^{\sqrt{1-x^2}} xy\sqrt{x^2+y^2}\, dydx \ .$$

Problem 5: Find the center of mass of the cardioid $r = 1 + \cos\theta$ if the density is the distance to the origin.

Problem 6: Evaluate the integral

$$\iiint_T x^2 y^2 z^2 dV \ ,$$

where T is the solid bounded by the planes $z = y + 1$, $y + z = 1$, $x = 0$, $x = 1$, $z = 0$.

Problem 7: Find the volume of the solid bounded above by the plane $z = y$ and below by the paraboloid $z = x^2 + y^2$. Hint: use cylindrical coordinates.

Problem 8: Find the centroid of the solid in Problem 7.

Problem 9: Find the volume of the solid bounded above by the cone $z^2 = x^2 + y^2$, below by the xy-plane and on the sides by the hemisphere $z = \sqrt{4 - x^2 - y^2}$. Hint: use spherical coordinates.

Problem 10: Evaluate the integral

$$\iint_\Omega \sin(x-y)\cos(x+2y)dA \ ,$$

where Ω is the parallelogram bounded by lines $x-y=0$, $x-y=\pi$, $x+2y=0$, $x+2y=\pi/2$.

Problem 11: Evaluate the integral

$$\iint_\Omega x^5 dA \ ,$$

where Ω is bounded by lines $y=x^3$, $y=1+x^3$, $y=2-x^3$, $y=3-x^3$.

Problem 12: Find the centroid of the hemisphere of radius r if the density is equal to the distance from the axis of symmetry of the hemisphere.

Problem 13: Evaluate:

$$\iiint_H z^3\sqrt{x^2+y^2+z^2}dV \ ,$$

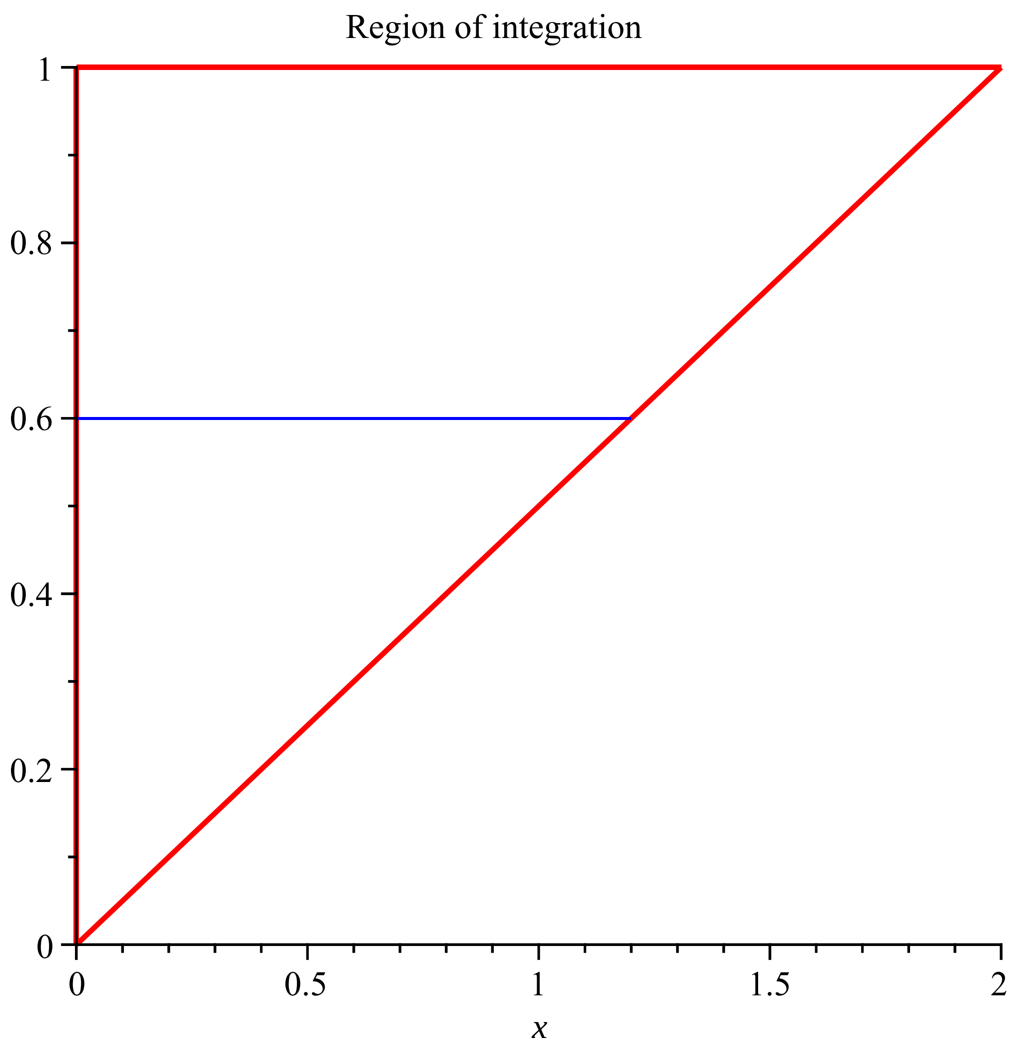
where H is the solid hemisphere that lies above xy-plane and has center at the origin and radius 1.

Problem 14: Find the center of mass of the solid bounded by the spheres $x^2+y^2+z^2=y$ and $x^2+y^2+z^2=2y$, if the density at a point is proportional to its distance from the origin.

MATH 265 / MAST 219
Exercise problems for the Midterm
Solutions:

Problem 1: The region of integrations is

```
> with(plots):
> pl1:=plot([1,x/2],x=0..2,color=red,thickness=2):
> pl2:=pointplot([[0,0],[0,1]],color=red,thickness=2,
> connect=true):
> pl3:=pointplot([[0,0.6],[1.2,0.6]],color=blue,thickness=1,
> connect=true):
> display(pl1,pl2,pl3,title="Region of integration");
```

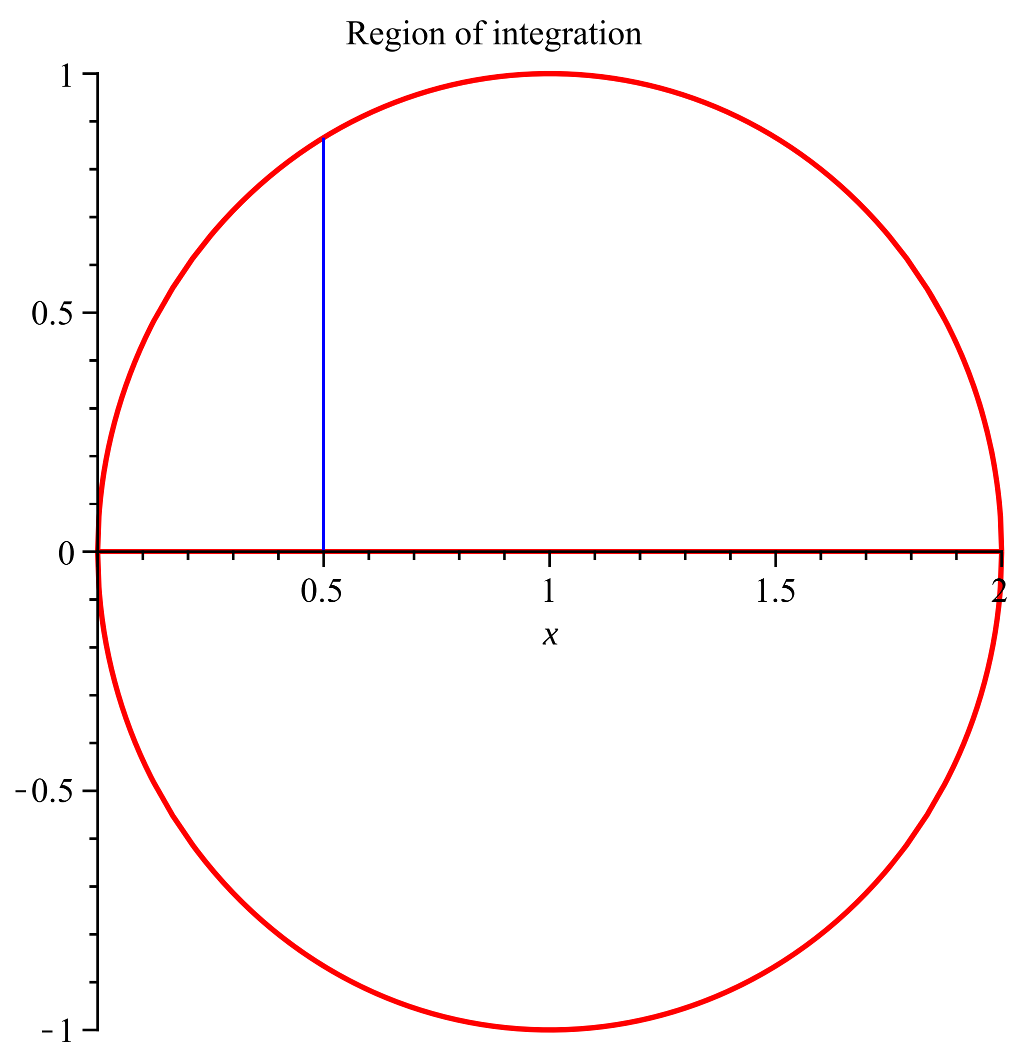


We have to integrate in dx first since $e^{-\frac{y^2}{2}}$ is not explicitelly integrable in dy. We have

$$\iint_{\Omega} e^{-\frac{y^2}{2}}\, dA = \int_0^1 \int_0^{2y} e^{-\frac{y^2}{2}}\, dx\, dy = \int_0^1 2\, y\, e^{-\frac{y^2}{2}}\, dy = \left[-2\, e^{-\frac{y^2}{2}} \right]_0^1 = -2e^{-\frac{1}{2}} + 2 = 2\left(1 - \frac{1}{\sqrt{e}}\right)$$

Problem 2: The floor of the solid is the disk $(x-1)^2 + y^2 \leq 1$ in the xy-plane (z = 0). The region of integration is

```
> with(plots):
> pl1:=plot([0,sqrt(1-(x-1)^2),-sqrt(1-(x-1)^2)],x=0..2,color=red,
  thickness=2):
> pl3:=pointplot([[0.5,0],[0.5,sqrt(1-(0.5-1)^2)]],color=blue,
  thickness=1,
> connect=true):
> display(pl1,pl3,title="Region of integration");
```



The "roof" is the plane $z = 2\,x + 2$. The solid is symmetric with respect to the x-axis so we can calculate the volume as

$$V=2\int_0^2\int_0^{\sqrt{1-(x-1)^2}}(2x+2)\,dydx=2\int_0^2\sqrt{1-(x-1)^2}\,(2x+2)\,dx=\Big(\text{by substitution :}$$
$$x-1=\sin u\,,\ dx=\cos u\,du\Big)=$$
$$=2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sqrt{1-\sin^2 u}\,(2\sin u+4)\cos u\,du=4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^2 u\,(\sin u+2)du=(\textit{since}\ \sin u\ \textit{is odd}$$
$$\textbf{and}\ \cos u\ \textit{even})=$$
$$4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}2\cos^2 u\,du=16\int_0^{\frac{\pi}{2}}\frac{1}{2}(1+\cos 2u)\,du=8\left[u+\frac{1}{2}\sin 2u\right]_0^{\frac{\pi}{2}}=4\pi$$

Problem 3: The base of the solid is the same as in the previous problem. The roof is $z=2-\sqrt{x^2+y^2}$. In polar coordinates the boundary of the base can be represented as $(r\cos\theta-1)^2+r^2\sin^2\theta=1$ or $r^2-2r\cos\theta=0$ or $r=2\cos\theta$.
The volume is (again by symmetry)

$$V=2\int_0^{\frac{\pi}{2}}\int_0^{2\cos\theta}(2-r)r\,dr\,d\theta=2\int_0^{\frac{\pi}{2}}\left[r^2-\frac{1}{3}r^3\right]_0^{2\cos\theta}d\theta=8\int_0^{\frac{\pi}{2}}\left(\cos^2\theta-\frac{2}{3}\cos^3\theta\right)d\theta$$

$$=8\int_0^{\frac{\pi}{2}}\left[\frac{1}{2}(1+\cos 2\theta)-\frac{2}{3}\left(1-\sin^2\theta\right)\cos\theta\right]d\theta=8$$

$$\left[\left(\frac{1}{2}\right)\theta+\frac{1}{4}\sin 2\theta-\frac{2}{3}\sin\theta+\frac{2}{9}\sin^3\theta\right]_0^{\frac{\pi}{2}}=$$

$$=8\left(\frac{\pi}{4}-\frac{2}{3}+\frac{2}{9}\right)=2\pi-\frac{32}{9}=2.727629752$$

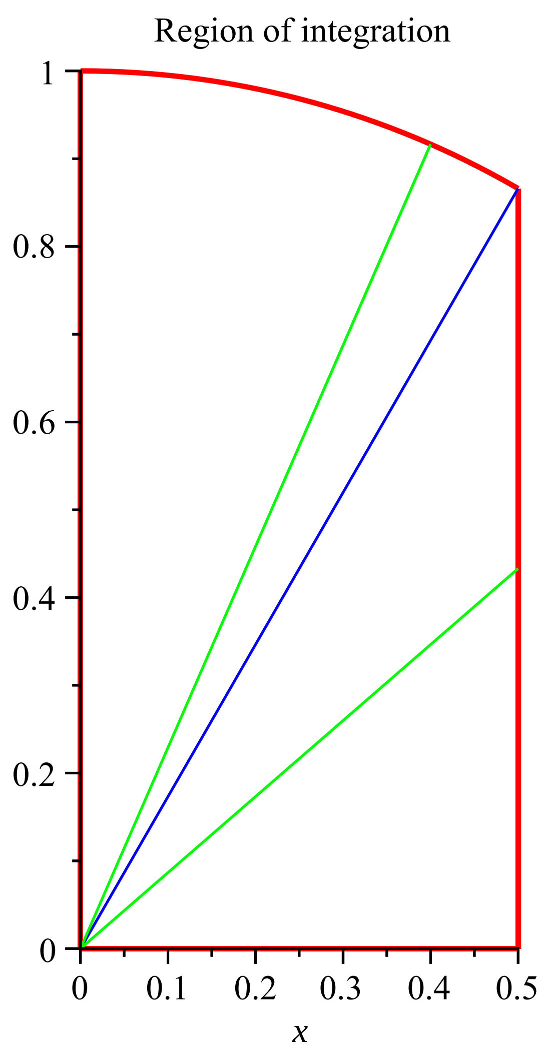
```
> 8*((1/4)*Pi-2/3+2/9);
  evalf(8*((1/4)*Pi-2/3+2/9));
```

$$2\pi-\frac{32}{9}$$

$$2.727629752 \qquad (1)$$

Problem 4: The integral contains expressions which may simplify in polar coordinates. The region of integration is a part of the unit disk:

```
> with(plots):
> pl1:=plot([0,sqrt(1-x^2)],x=0..1/2,color=red,thickness=2):
> pl2:=pointplot([[0,0],[0,1]],color=red,thickness=2,connect=true)
  :
  pl2a:=pointplot([[1/2,0],[1/2,sqrt(3/4)]],color=red,thickness=2,
  connect=true):
> pl3:=pointplot([[0,0],[1/2,sqrt(3/4)]],color=blue,thickness=1,
  connect=true):
  pl3a:=pointplot([[0,0],[1/2,sqrt(3/4)/2]],color=green,thickness=
  1,connect=true):
  pl3b:=pointplot([[0,0],[0.4,sqrt(1-0.4^2)]],color=green,
  thickness=1,connect=true):
> display(pl1,pl2,pl2a,pl3,pl3a,pl3b,title="Region of
  integration");
```



We have to consider two parts of it:

First, the part with a vertical right hand side: thr upper vertex is $\left(\frac{1}{2}, \sqrt{\frac{3}{4}}\right)$ so the corresponding θ has

tan θ =$\sqrt{3}$ which means that $\theta = \frac{\pi}{3}$. The points on the right hand side satisfy

$$\frac{\frac{1}{2}}{r} = \cos\theta \ \textbf{ or } \ r = \frac{1}{2\cos\theta} .$$

We have

$$\int_0^{\frac{1}{2}} \int_0^{\sqrt{1-x^2}} xy\sqrt{x^2+y^2}\, dy\, dx = \int_0^{\frac{\pi}{3}} \int_0^{\frac{1}{2\cos\theta}} r^3 \sin\theta\cos\theta\, r\, dr\, d\theta + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \int_0^1 r^3 \sin\theta\cos\theta\, r\, dr\, d\theta$$

The first part: the inner integral is $\int_0^{\frac{1}{2\cos\theta}} r^4\, dr = \frac{1}{5}\ \frac{1}{32\cos^5\theta}$ Then, $\frac{1}{160}\int_0^{\frac{\pi}{3}} \sin\theta\ \frac{1}{\cos^4\theta}\, d\theta =$

$$= \frac{1}{160}\left[\frac{1}{3}\ \frac{1}{\cos^3\theta}\right]_0^{\frac{\pi}{3}} = \frac{1}{160}\ \frac{1}{3}[8-1] = \frac{7}{3\cdot 160}$$

The second part is equal to $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin\theta\cos\theta\, d\theta \int_0^1 r^4 dr = \left[\frac{1}{2}\sin^2\theta\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{5} = \frac{1}{10}\left[1-\frac{3}{4}\right] = \frac{1}{40}$

Answer: $\frac{7}{3\cdot 160} + \frac{1}{40}$

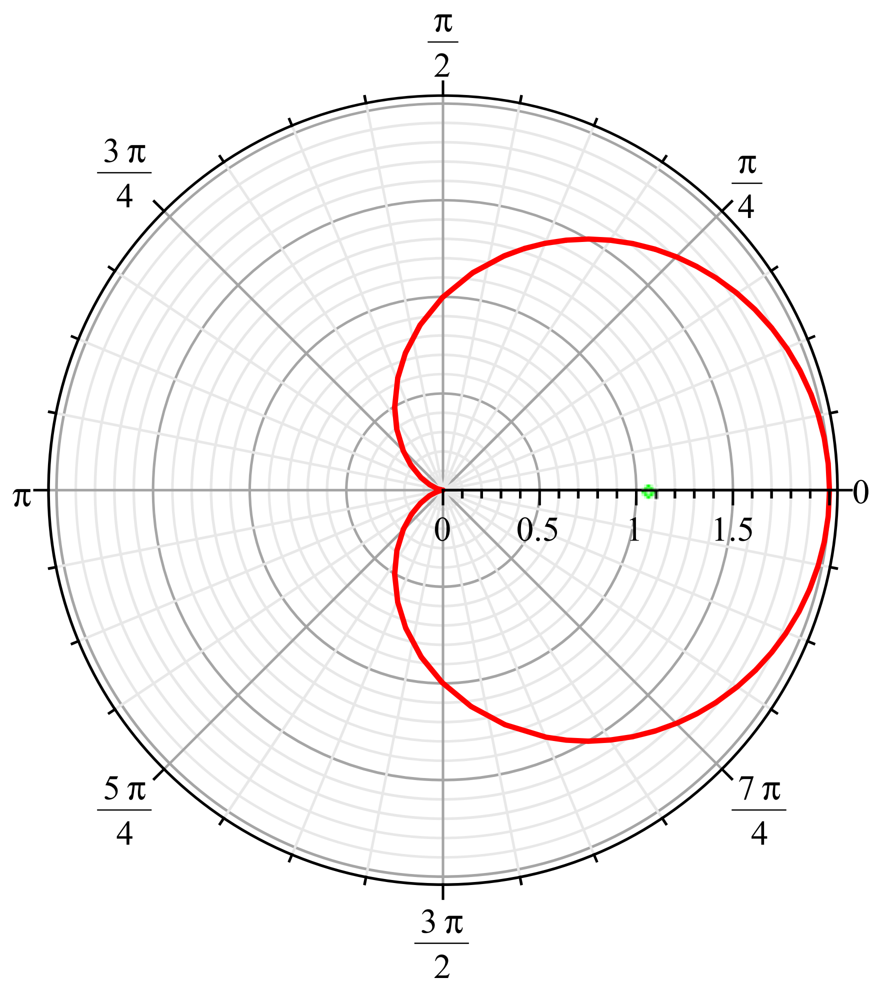
```
> 7/(3*160)+1/40;
```

$$\frac{19}{480} \quad (2)$$

Problem 5: The cardioid is shown below

```
> with(plots):
> pl1:=polarplot(1+cos(th),th=0..2*Pi,color=red,thickness=2):
  pl2:=pointplot([[21/20,0]],color=green,thickness=3):
> display(pl1,pl2,title="Region of integration");
```

Region of integration



We have to calculate the mass $m = \iint_{cardioid} \rho(x, y)\, dA$ and the moments $M_x = \iint_{cardioid} y\, \rho(x, y)\, dA$ and $M_y = \iint_{cardioid} x\, \rho(x, y)\, dA$. Using the polar coordinates we obtain

$$m = \int_0^{2\pi} \int_0^{1+\cos\theta} r r\, dr\, d\theta = \frac{1}{3} \int_0^{2\pi} [1 + \cos\theta]^3\, d\theta = \frac{1}{3} \left[\int_0^{2\pi} 1\, d\theta + 3 \int_0^{2\pi} \cos\theta\, d\theta + 3 \int_0^{2\pi} \cos^2\theta\, d\theta + \int_0^{2\pi} \cos^3\theta\, d\theta \right] =$$

$$= \frac{1}{3} \left[2\pi + 0 + 6 \int_0^{\pi} \frac{1}{2} (1 + \cos 2\theta)\, d\theta + 0 \right] = \frac{1}{3} \left[2\pi + 3 \left[\theta + \frac{1}{2} \sin 2\theta \right]_0^{\pi} \right] = \frac{1}{3} [5\pi] = \frac{5}{3} \pi$$

By symmetry, we have $M_x = 0$. *We have*

$$M_y = \int_0^{2\pi} \int_0^{1+\cos\theta} r \cos\theta\, r\, r\, dr\, d\theta = \frac{1}{4} \int_0^{2\pi} \cos\theta\, [1 + \cos\theta]^4\, d\theta =$$

$$= \frac{1}{4}\left[\int_0^{2\pi} \cos\theta\, d\theta + 4\int_0^{2\pi} \cos^2\theta\, d\theta + 6\int_0^{2\pi} \cos^3\theta\, d\theta + 4\int_0^{2\pi} \cos^4\theta\, d\theta + \int_0^{2\pi} \cos^5\theta\, d\theta\right]$$

Since

$$\cos^4\theta = \left(\frac{1}{2}(1+\cos 2\theta)\right)^2 = \frac{1}{4}(1+2\cos 2\theta+\cos^2 2\theta) = \frac{1}{4}\left(1+2\cos 2\theta+\frac{1}{2}(1+\cos 4\theta)\right)$$

$$= \frac{1}{8}(3+4\cos 2\theta+\cos 4\theta)$$

we have

$$M_y = \frac{1}{4}\left[\frac{4}{2}\int_0^{2\pi}(1+\cos 2\theta)\, d\theta + \frac{4}{8}\int_0^{2\pi}(3+4\cos 2\theta+\cos 4\theta)d\theta\right] = \frac{1}{4}\left[\frac{4}{2}2\pi+\frac{4}{8}3\cdot 2\pi\right]$$

$$= \frac{7}{4}\pi$$

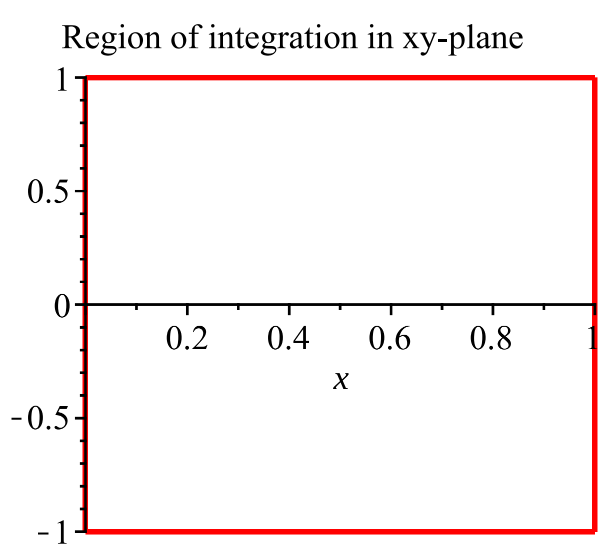
Thus, the center of mass is

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right) = \left(\frac{7}{4}\frac{3}{5}, 0\right) = \left(\frac{21}{20}, 0\right)$$ - green point on the picture

Problem 6: Evaluate the integral $\iiint_T x^2y^2z^2\, dV$, *where* T is bounded by the planes $z=y+1,\, y+z=1,\, x=0,\, x=1,\, z=0.$

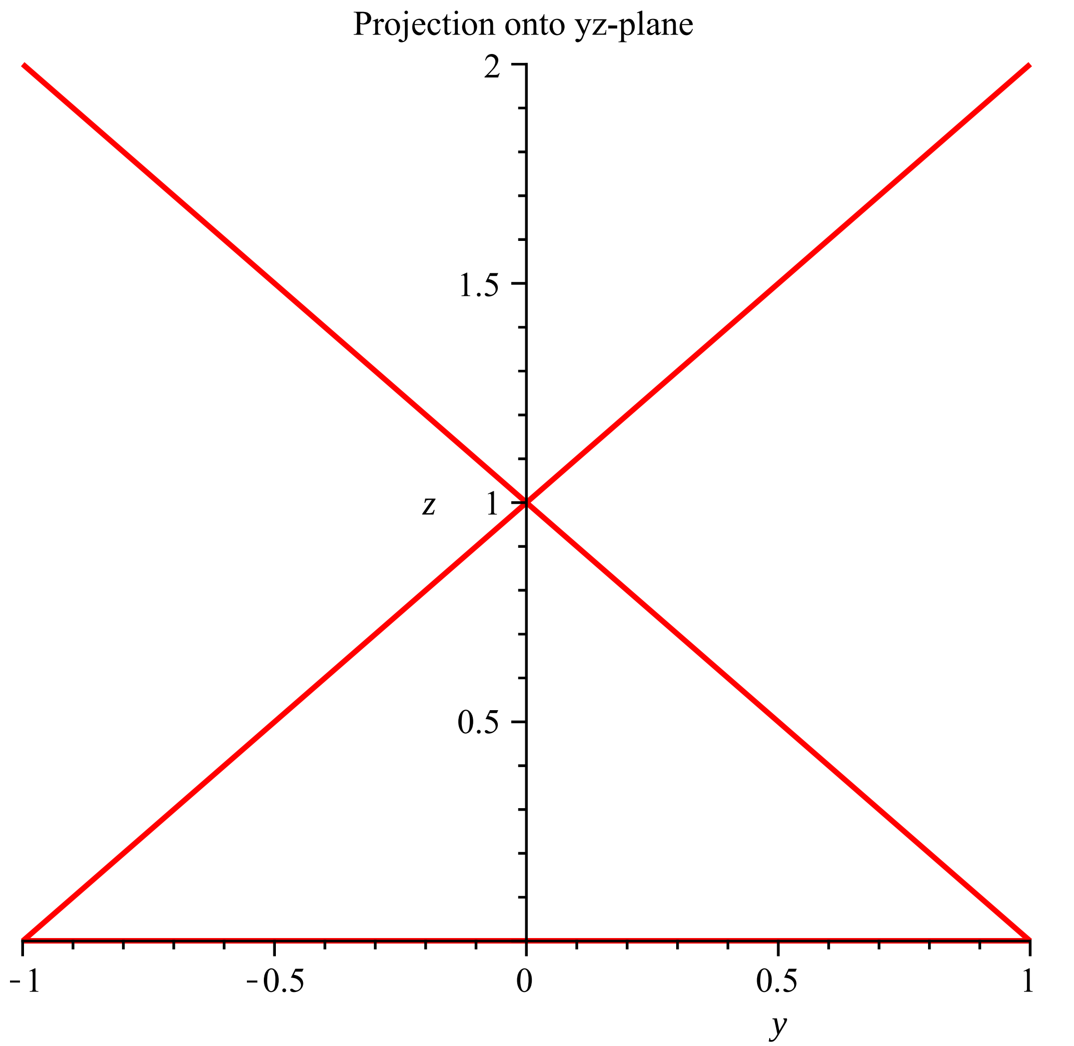
The "base" of the solid is in xy-plane and is bounded by the lines : $0=y+1,\, y+0=1,\, x=0,\, x=1.$

```
> with(plots):
> pl1:=plot([1,-1],x=0..1,color=red,thickness=2):
> pl2:=pointplot([[0,-1],[0,1]],color=red,thickness=2,connect=
  true):
  pl2a:=pointplot([[1,-1],[1,1]],color=red,thickness=2,connect=
  true):
> display(pl1,pl2,pl2a,title="Region of integration in xy-plane");
```



To easier see the "roof" of the solid, we project it onto yz-plane: The projection is bounded by the lines: $z=y+1,\ y+z=1,\ z=0.$

```
> with(plots):
> pl1:=plot([y+1,1-y,0],y=-1..1,color=red,thickness=2):
> display(pl1,title="Projection onto yz-plane",labels=[y,z]);
```



The projection is only the triangle at bottom of the picture. We see that we have to split the integral into 2 parts
as the "roof" (which looks as a real roof) consists of two planes. We have

$$\iiint_T x^2y^2z^2\,dV = \int_0^1\int_{-1}^0\int_0^{1+y} x^2y^2z^2\,dz\,dy\,dx \;+\; \int_0^1\int_0^1\int_0^{1-y} x^2y^2z^2\,dz\,dy\,dx =$$

$$= \int_0^1\int_{-1}^0 x^2y^2\ \frac{1}{3}\left[z^3\right]_0^{1+y}\,dy\,dx \;+\; \int_0^1\int_0^1 x^2y^2\ \frac{1}{3}\left[z^3\right]_0^{1-y}dy\,dx =$$

$$= \frac{1}{3}\left\{\int_0^1\int_{-1}^0 x^2y^2\ (1+y)^3\ dy\,dx \;+\; \int_0^1\int_0^1 x^2y^2\,(1-y)^3\,dy\,dx\right\} =$$

$$= \frac{1}{3}\left\{\int_0^1 x^2 dx \int_{-1}^{0} y^2 + 3\,y^3 + 3\,y^4 + y^5 \; dy \; + \int_0^1 x^2\, dx \int_0^1 y^2 - 3\,y^3 + 3\,y^4 - y^5 \; dy\right\} =$$

$$= \frac{1}{3}\left\{\frac{1}{3}\left(\frac{1}{3} - \frac{3}{4} + \frac{3}{5} - \frac{1}{6}\right) + \frac{1}{3}\left(\frac{1}{3} - \frac{3}{4} + \frac{3}{5} - \frac{1}{6}\right)\right\} = \frac{1}{270}$$

```
> (1/3)*((1/3)*(1/3-3/4+3/5-1/6)+(1/3)*(1/3-3/4+3/5-1/6));
```

$$\frac{1}{270} \qquad (3)$$

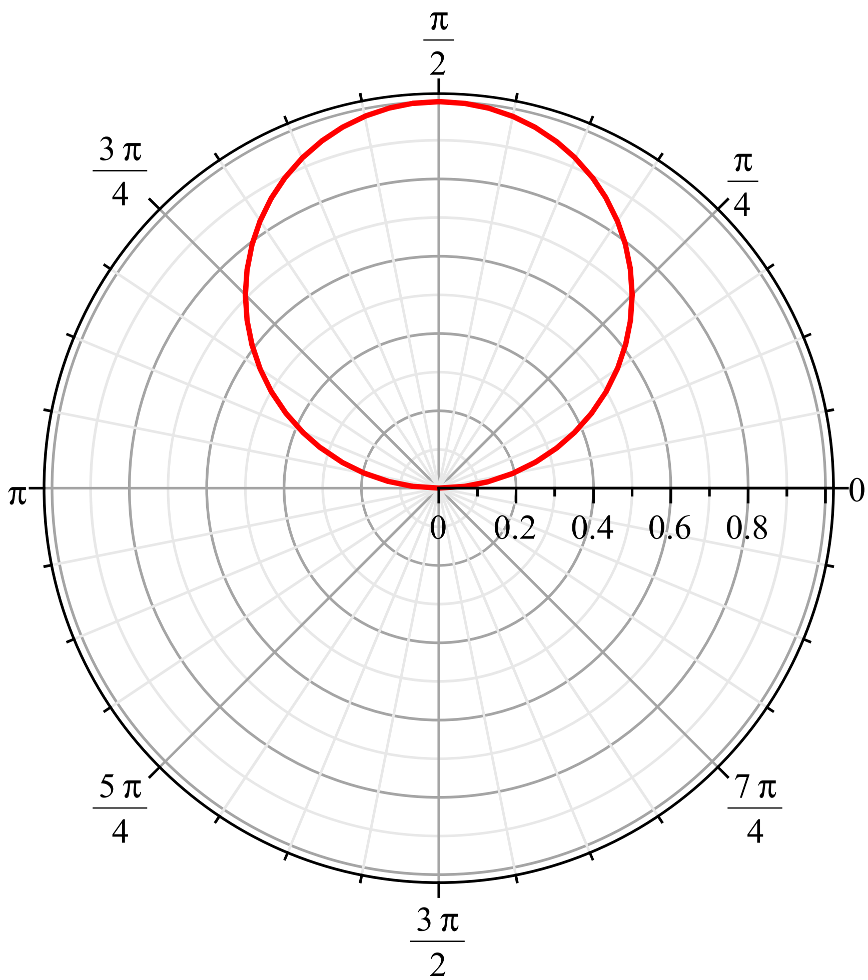
Problem 7: Find the volume of the solid bounded above by the plane z = y and below by the paraboloid $z = x^2 + y^2$. Hint: use cylindrical coordinates.
The projection (on z =0) of the intersection of these surfaces is the curve $y = x^2 + y^2$ or

$$\frac{1}{4} = x^2 + \left(y - \frac{1}{2}\right)^2$$

In polar coordinates this is : $r \sin\theta = r^2$ or $r = \sin\theta$, for ,$0 \le \theta \le \pi$.

```
> with(plots):
> pl1:=polarplot(sin(th),th=0..Pi,color=red,thickness=2):
> display(pl1,title="Region of integration");
```

Region of integration



The volume is equal to $V=\int_0^{\pi}\int_0^{\sin\theta}\left(r\sin\theta-r^2\right)\ r\,dr\,d\theta=\int_0^{\pi}\left[\frac{1}{3}r^3\sin\theta-\frac{1}{4}r^4\right]_0^{\sin\theta}d\theta=$

$$=\int_0^{\pi}\left[\frac{1}{3}\sin^4\theta-\frac{1}{4}\sin^4\theta\right]d\theta=\frac{1}{12}\int_0^{\pi}\sin^4\theta\,d\theta=\frac{1}{12}\,\frac{1}{2}\,\frac{1}{2}\int_0^{\pi}\left(1-\cos2\theta\right)^2d\theta=\frac{1}{48}\int_0^{\pi}\left(1-2\cos2\theta+\cos^2 2\theta\right)d\theta=$$

$=$ since the middle term integrates to $0=\frac{1}{48}\int_0^{\pi}\left(1+\frac{1}{2}\left(1+\cos4\theta\right)\right)d\theta=\frac{1}{48}\int_0^{\pi}\frac{3}{2}d\theta=\frac{1}{32}\pi$

Problem 8: Find the centroid of the solid in Problem 7.
Since the density is uniform the mass is proportional to the volume and we can assume that
$m=\frac{1}{32}\pi.\left(\text{assuming }\rho=1\right)$
We have to calculate the moments:
$M_{yz}=\iiint_T x\,dV$, $M_{xz}=\iiint_T y\,dV$, $M_{xy}=\iiint_T z\,dV$. By symmetry the moment $M_{yz}=0$.

We have $M_{xz}=\int_0^{\pi}\int_0^{\sin\theta}\int_{r^2}^{r\sin\theta}\left(r\sin\theta\right)dz\,r\,dr\,d\theta=\int_0^{\pi}\int_0^{\sin\theta}\left(r\sin\theta\right)\left(r\sin\theta-r^2\right)\ r\,dr\,d\theta=$

$$=\int_0^{\pi}\left[\frac{1}{4}r^4\sin^2\theta-\frac{1}{5}r^5\sin\theta\right]_0^{\sin\theta}d\theta=\int_0^{\pi}\left[\frac{1}{4}\sin^6\theta-\frac{1}{5}\sin^6\theta\right]d\theta=\frac{1}{20}\int_0^{\pi}\sin^6\theta\,d\theta$$

We have $\sin^6\theta=\frac{1}{8}\left(1-\cos2\theta\right)^3=\frac{1}{8}\left(1-3\cos2\theta+3\cos^2 2\theta-\cos^3 2\theta\right)$. Thus,

$$M_{xz}=\frac{1}{20}\,\frac{1}{8}\int_0^{\pi}\left(1+3\cos^2 2\theta\right)d\theta=\frac{1}{160}\int_0^{\pi}\left(1+\frac{3}{2}\left(1+\cos4\theta\right)\right)d\theta=\frac{1}{160}\int_0^{\pi}\left(\frac{5}{2}\right)d\theta=\frac{1}{64}\pi$$

We have $M_{xy}=\int_0^{\pi}\int_0^{\sin\theta}\int_{r^2}^{r\sin\theta}z\ dz\,r\,dr\,d\theta=\int_0^{\pi}\int_0^{\sin\theta}\frac{1}{2}\left[r^2\sin^2\theta-r^4\right]\ r\,dr\,d\theta=$

$$=\frac{1}{2}\int_0^{\pi}\left[\frac{1}{4}r^4\sin^2\theta-\frac{1}{6}r^6\right]_0^{\sin\theta}d\theta=\frac{1}{2}\int_0^{\pi}\left[\frac{1}{4}\sin^6\theta-\frac{1}{6}\sin^6\theta\right]d\theta=\frac{1}{24}\int_0^{\pi}\sin^6\theta\,d\theta=$$

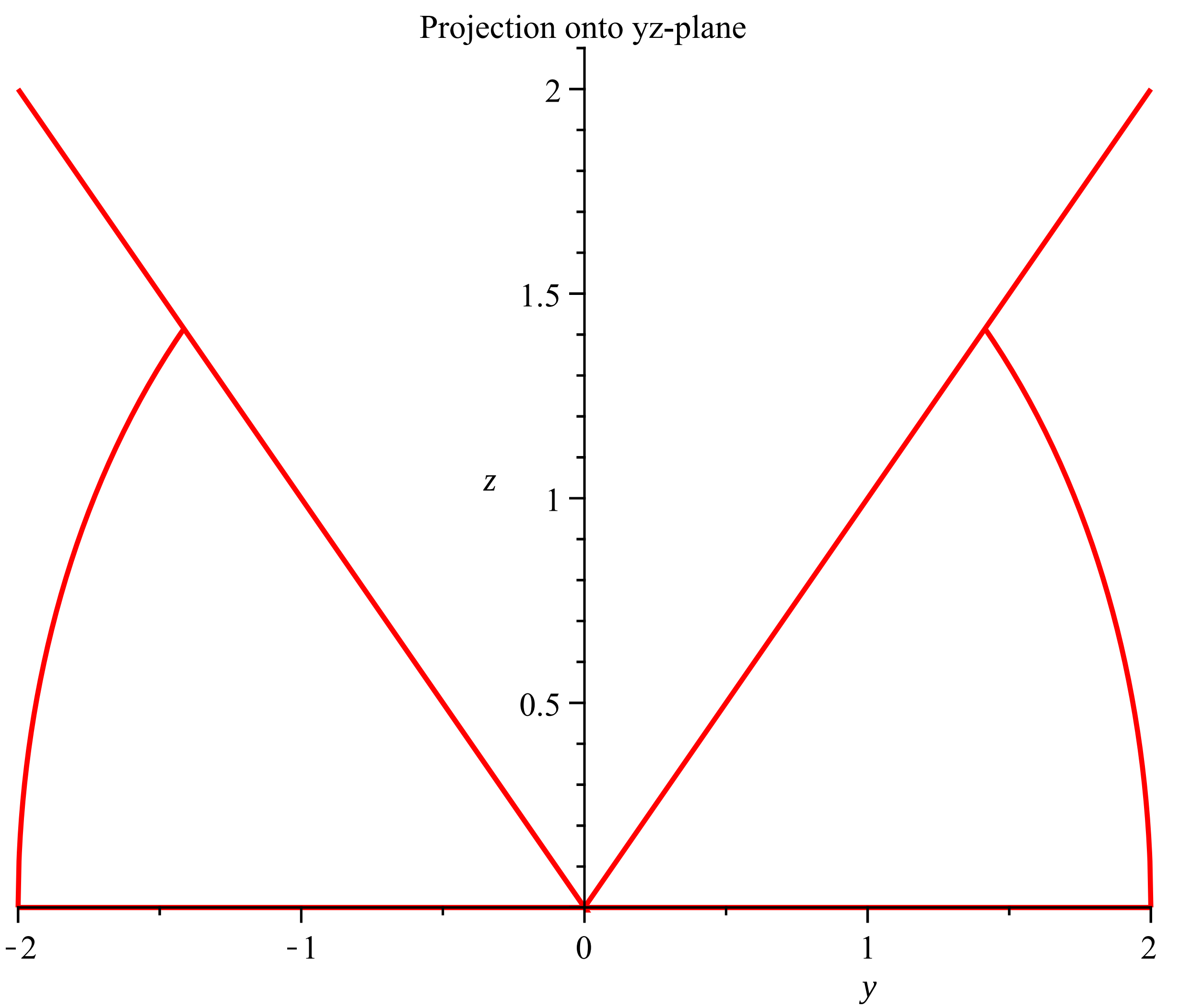
$$=\frac{20}{24}\,\frac{1}{64}\pi=\frac{5}{384}\pi$$

The centroid is $\left(\bar{x},\bar{y},\bar{z}\right)=\frac{1}{m}\left(M_{yz}\ ,\ M_{xz}\ ,M_{xy}\right)=\frac{32}{\pi}\left(0,\frac{\pi}{64},\frac{5}{384}\pi\right)=\left(0,\frac{1}{2},\frac{5}{12}\right)$

Problem 9: Find the volume of the solid bounded above by the cone $z^2=x^2+y^2$,
below by the xy-plane and on the sides by the hemisphere $z=\sqrt{4-x^2-y^2}$. Hint: use spherical coordinates.

The intersection with zy-plane is bounded by the curves $z^2 = y^2$ or $z = \pm y$ **and** $z = \sqrt{4 - y^2}$ or $z^2 + y^2 = 4$ *with* $z \geq 0$. Our solid is obtained by rotating this intersection around the z-axis.

```
> with(plots):
> pl1:=plot([y,-y,0],y=-2..2,z=0..2.1,color=red,thickness=2):
  pl2:=plot([sqrt(4-y^2)],y=-2..-sqrt(2),z=0..2.1,color=red,
  thickness=2):
  pl3:=plot([sqrt(4-y^2)],y=sqrt(2)..2,z=0..2.1,color=red,
  thickness=2):
> display(pl1,pl2,pl3,title="Projection onto yz-plane",labels=[y,
  z]);
```



The angle between the the base and the lines $z = \pm y$ is $\pi/4$. In spherical coordinates our solid is a rectangular box:

$B = \left\{ (\rho, \theta, \phi) : 0 \leq \rho \leq 2, 0 \leq \theta \leq 2\pi, \frac{\pi}{4} \leq \phi \leq \frac{\pi}{2} \right\}$ and its volume is

$$V=\int_0^{2\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^2 \rho^2 \sin\phi \; d\rho\, d\phi\, d\theta = \int_0^{2\pi} d\theta \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin\phi \, d\phi \int_0^2 \rho^2 \; d\rho = 2\pi\,[-\cos\phi]_{\frac{\pi}{4}}^{\frac{\pi}{2}} \; \frac{1}{3}\left[\rho^3\right]_0^2 =$$

$$=2\pi \, \frac{\sqrt{2}}{2} \; \frac{1}{3} 8 = \frac{8\sqrt{2}\,\pi}{3}$$

Problem 10: Evaluate the integral

$$\iint_{\Omega} \sin(x-y)\cos(x+2y)\; dA,$$

where Ω is the parallelogram bounded by lines $x-y=0$, $x-y=\pi$, $x+2y=0$, $x+2y=\frac{\pi}{2}$.

We introduce new variables $u=x-y$ and $v=x+2y$. In these variables the region of integration becomes

a rectangle $R=\left\{(u,v) : 0 \le u \le \pi,\, 0 \le v \le \frac{\pi}{2}\right\}$.

Solving the system $\begin{cases} u=x-y \\ v=x+2y \end{cases}$ for x and y we obtain: $\begin{cases} x=\frac{(2u+v)}{3} \\ y=\frac{(v-u)}{3} \end{cases}$

Thus, the Jacobian $\frac{\partial(x,y)}{\partial(u,v)} = det\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = det\begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{bmatrix} = \frac{3}{9}$ **and** $\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{3}$

We have

$$\iint_{\Omega} \sin(x-y)\cos(x+2y)\; dA = \int_0^{\pi}\int_0^{\frac{\pi}{2}} \sin u \cos v \;\; \frac{1}{3}\; dv\, du = \frac{1}{3}\int_0^{\pi} \sin u \;\; du \int_0^{\frac{\pi}{2}} \cos v \;\; dv =$$

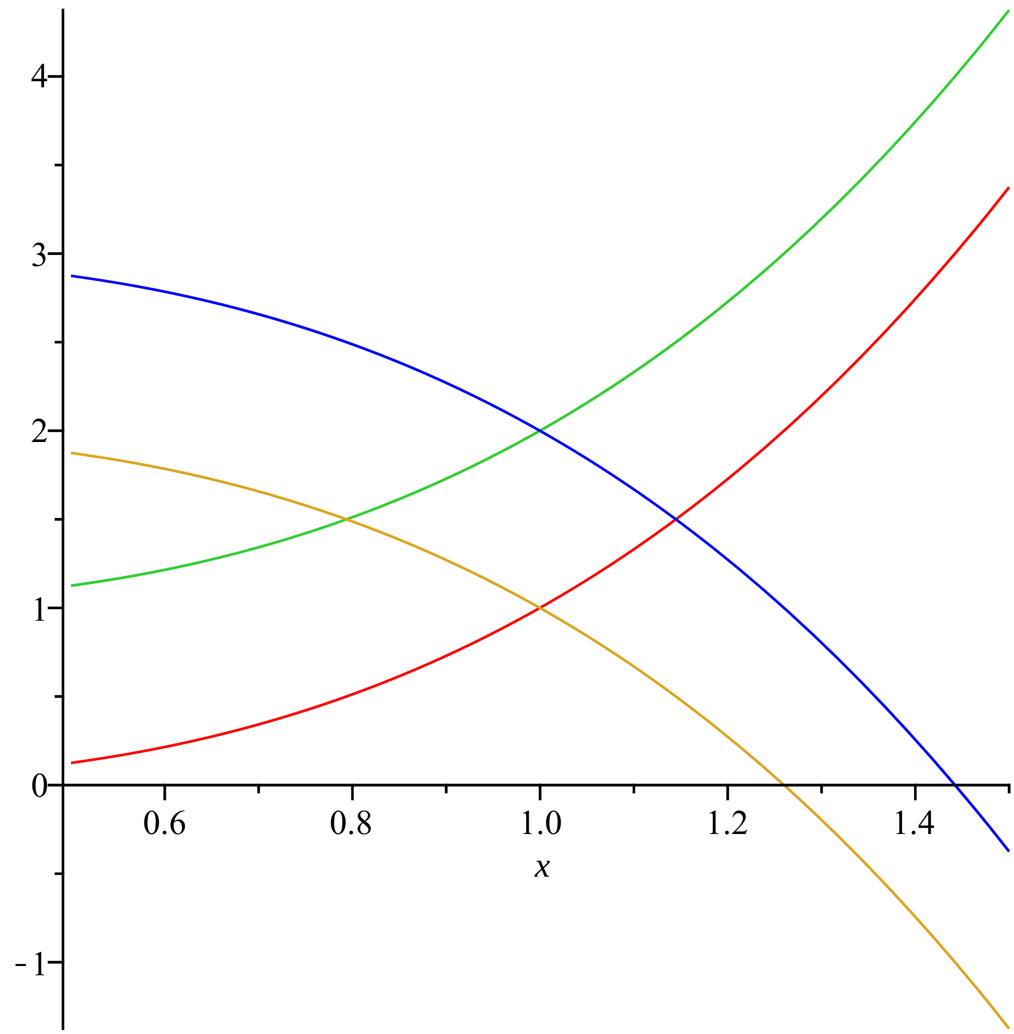
$$= \frac{1}{3}\,[-\cos u]_0^{\pi}[\sin v]_0^{\frac{\pi}{2}} = \frac{1}{3} 2\cdot 1 = \frac{2}{3}.$$

Problem 12: Evaluate the integral

$$\iint_{\Omega} x^5 dA,$$

where Ω is bounded by lines $y=x^3$, $y=1+x^3$, $y=2-x^3$, $y=3-x^3$.

```
> plot([x^3,1+x^3,2-x^3,3-x^3],x=0.5..1.5);
```



We introduce new variables $u = y - x^3$ and $v = x^3 + y$. In these variables the region of integration becomes
a rectangle $R = \{(u, v) : 0 \le u \le 1, 2 \le v \le 3\}$.

The Jacobian $\frac{\partial(u, v)}{\partial(x, y)} = det \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} = det \begin{bmatrix} -3x^2 & 1 \\ 3x^2 & 1 \end{bmatrix} = -6x^2$ **and** $\left| \frac{\partial(u, v)}{\partial(x, y)} \right| = 6x^2$

We can calculate : $v - u = 2x^3$ so $x^2 = \left(\frac{v - u}{2} \right)^{\frac{2}{3}} = \frac{1}{\sqrt[3]{4}} (v - u)^{\frac{2}{3}}$

Thus, the Jacobian $\left| \frac{\partial(x, y)}{\partial(u, v)} \right| = \left| \frac{\partial(u, v)}{\partial(x, y)} \right|^{-1} = \frac{\sqrt[3]{4}}{6(v - u)^{\frac{2}{3}}}$

We have $\iint_{\Omega} x^5 dA$

$$= \int_2^3\int_0^1 \left(\frac{v-u}{2}\right)^{\frac{5}{3}} \frac{\sqrt[3]{4}}{6\,(v-u)^{\frac{2}{3}}}\, du\, dv = \frac{1}{2^{\frac{5}{3}}}\,\frac{2^{\frac{2}{3}}}{6}\int_2^3\int_0^1 (v-u)\,du\,dv = \frac{1}{12}\int_2^3 \left(v-\frac{1}{2}\right)dv =$$

$$= \frac{1}{12}\left[\frac{v^2}{2}-\frac{v}{2}\right]_2^3 = \frac{1}{24}[9-3-4+2] = \frac{1}{6}$$

Problem 12: Find the centroid of the hemisphere of radius r if the density is equal to the distance from the axis of symmetry of the hemisphere.

We can assume that the hemisphere is the upper half of the sphere centered at the origin. The z-axis is the axis of the symmetry.

The distance of the point (x,y,z) from the z-zxis is $\sqrt{x^2+y^2}$.

We will use the spherical coordinates $x = \rho\cos\theta\sin\phi$, $y = \rho\sin\theta\sin\phi$, $z = \rho\cos\phi$.

Then, the density is $\rho(x,y,z) = \sqrt{x^2+y^2} = \sqrt{\rho^2\cos^2\theta\sin^2\phi + \rho^2\sin^2\theta\sin^2\phi} = \rho\sin\phi$

First we will find the mass: $m = \iiint_H \sqrt{x^2+y^2}\, dV = \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^r \rho\sin\phi\,\rho^2\sin\phi\, d\rho\, d\phi\, d\theta =$

$$= 2\pi\int_0^{\frac{\pi}{2}} \frac{1}{2}(1-\cos 2\phi)\, d\phi \left[\frac{1}{4}\rho^4\right]_0^r = 2\pi\,\frac{\pi}{4}\,\frac{1}{4}r^4 = \frac{1}{8}\pi^2 r^4$$

We have to calculate the moments:

$M_{yz} = \iiint_H x\, dV$, $M_{xz} = \iiint_H y\, dV$, $M_{xy} = \iiint_H z\, dV$. The moments M_{yz} and M_{xz} *are* 0 by symmetry.

We have

$$M_{xy} = \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^r \rho\cos\phi\ \rho\sin\phi\,\rho^2\sin\phi\, d\rho\, d\phi\, d\theta = 2\pi\int_0^{\frac{\pi}{2}} \cos\phi\ \sin^2\phi\, d\phi\left[\frac{1}{5}\rho^5\right]_0^r =$$

$$= 2\pi\ \frac{1}{3}\left[\sin^3\phi\right]_0^{\frac{\pi}{2}}\ \frac{1}{5}r^5 = \frac{2}{15}\pi r^5$$

We have $(\bar{x}, \bar{y}, \bar{z}) = \frac{1}{m}(M_{yz}, M_{xz}, M_{xy}) = \frac{8}{\pi^2 r^4}\left(0, 0, \frac{2}{15}\pi r^5\right) = \left(0, 0, \frac{16}{15\pi} r\right)$

In general we can say that the center of the mass of the hemisphere of radius r with this density lies on the axis of symmetry

at the distance $\frac{16}{15\pi}$ *r* from the base.

Problem 13: Evaluate:

$$\iiint_H z^3\sqrt{x^2+y^2+z^2}\, dV,$$

where H is the solid hemisphere that lies above xy-plane and has center at the origin and radius 1.

We have

$$\iiint_H z^3\sqrt{x^2+y^2+z^2}\,dV = \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^1 \rho^3\cos^3\phi\cdot\rho\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = 2\pi\int_0^1 \rho^6 d\rho\int_0^{\frac{\pi}{2}}\cos^3\phi\sin\phi\,d\phi =$$

$$= 2\pi\ \frac{1}{7}\ \left[-\frac{1}{4}\cos^4\phi\right]_0^{\frac{\pi}{2}} = 2\pi\ \frac{1}{7}\ \frac{1}{4}\ [0+1] = \frac{\pi}{14}$$

Problem 14:

Find the center of mass of the solid bounded by the spheres $x^2+y^2+z^2=y$ and $x^2+y^2+z^2=2y$ if the density at a point is proportional to its distance from the origin.

The spheres are: $x^2+y^2-y+\frac{1}{4}+z^2=\frac{1}{4}$ **or** $x^2+\left(y-\frac{1}{2}\right)^2+z^2=\left(\frac{1}{2}\right)^2$

and $x^2+y^2-2y+1+z^2=1$ **or** $x^2+(y-1)^2+z^2=1$

In spherical coordinates they are $\rho^2=\rho\sin\phi\sin\theta$ **or** $\rho=\sin\phi\sin\theta$

and $\rho=2\sin\phi\sin\theta$

The density is $\varrho(\rho,\varphi,\theta)=\rho$ (*We can assume that the constant is* $K=1$). We have

$$m=\int_0^{\pi}\int_0^{\pi}\int_{\sin\phi\sin\theta}^{2\sin\phi\sin\theta}\rho\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta=\int_0^{\pi}\int_0^{\pi}\sin\phi\left[\frac{1}{4}\rho^4\right]_{\sin\phi\sin\theta}^{2\sin\phi\sin\theta}d\phi\,d\theta=\frac{1}{4}\int_0^{\pi}\int_0^{\pi}\sin\phi[16$$
$$-1]\sin^4\phi\sin^4\theta\,d\phi\,d\theta=$$

$$=\frac{15}{4}\int_0^{\pi}\sin^5\phi\,d\phi\int_0^{\pi}\sin^4\theta\,d\theta$$

We have $\sin^4\alpha=\left(1-\cos^2\alpha\right)^2=1-2\cos^2\alpha+\cos^4\alpha$ so we have

$$\int_0^{\pi}\sin^5\phi\,d\phi=\int_0^{\pi}\left(\sin\phi-2\sin\phi\cos^2\phi+\sin\phi\cos^4\phi\right)d\phi=\left[-\cos\phi+\frac{2}{3}\cos^3\phi-\frac{1}{5}\cos^5\phi\right]_0^{\pi}=1+1$$
$$-\frac{2}{3}-\frac{2}{3}+\frac{1}{5}+\frac{1}{5}=\frac{16}{15}$$

```
> 1+1-2/3-2/3+1/5+1/5;
```

$$\frac{16}{15} \qquad (4)$$

and $\int_0^{\pi}\sin^4\theta\,d\theta=\int_0^{\pi}\left(1-2\cos^2\theta+\cos^4\theta\right)d\theta=\int_0^{\pi}\left(1-2\,\frac{1+\cos 2\theta}{2}+\left(\frac{1+\cos 2\theta}{2}\right)^2\right)d\theta=$

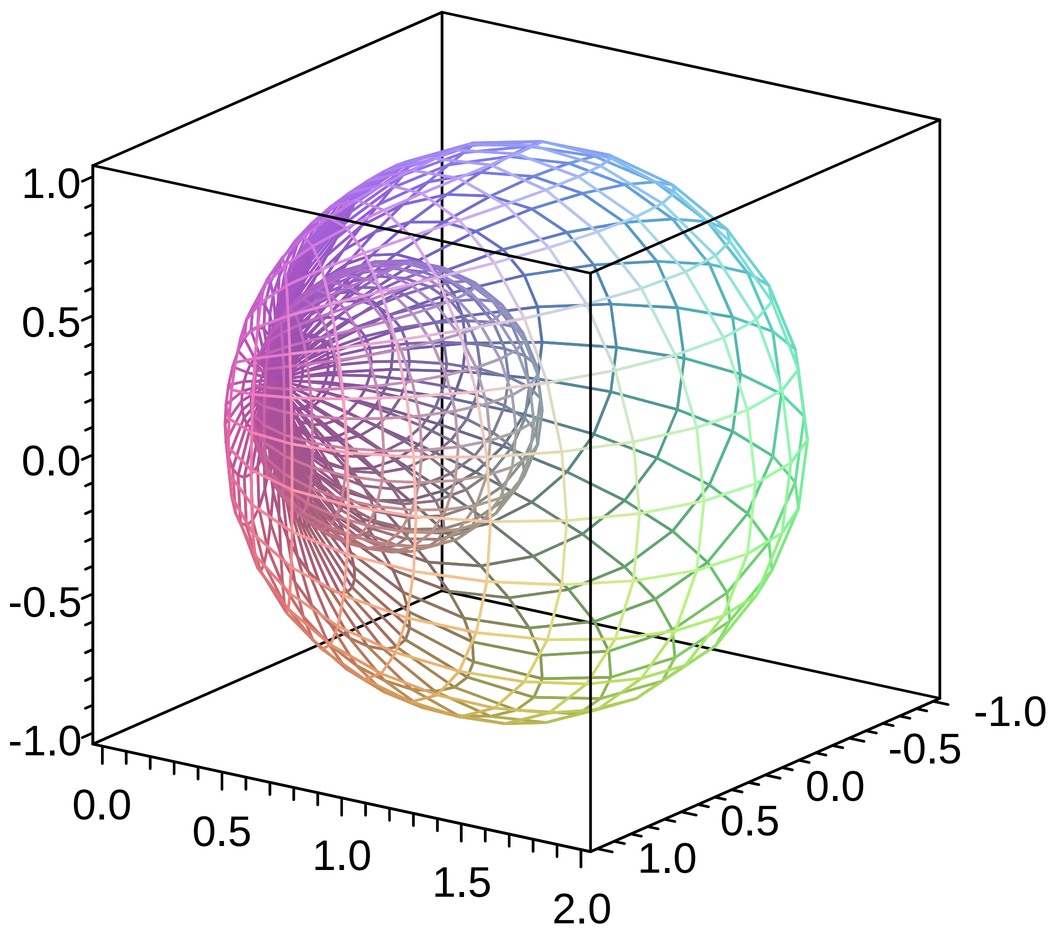
$$=\int_0^{\pi}\left(1-2\,\frac{1+\cos 2\theta}{2}+\frac{1}{4}\left(1+2\cos 2\theta+\cos^2 2\theta\right)\right)d\theta=$$

$$= \int_0^{\pi} \left(-\cos 2\,\theta + \frac{1}{4} \left(1 + 2\cos 2\,\theta + \frac{1 + \cos 4\,\theta}{2} \right) \right) d\theta = \frac{\pi}{4} + \frac{\pi}{8} = \frac{3\,\pi}{8}$$

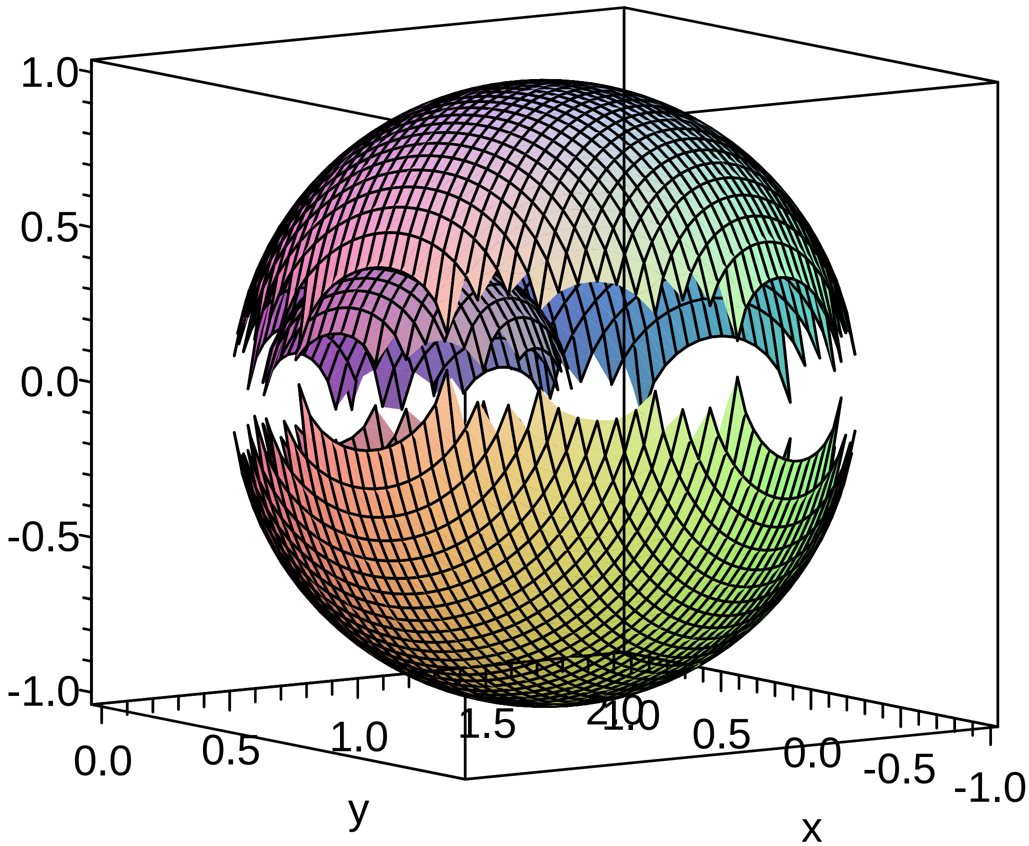
We used the fact that $\int_0^{\pi} \cos 2\,\theta \, d\theta = \int_0^{\pi} \cos 4\,\theta \, d\theta = 0$

Thus, $m = \frac{15}{4} \frac{16}{15} \quad \frac{3\,\pi}{8} = \frac{3\,\pi}{2}$

\> *plot3d*([sin(ϕ) ·sin(θ), 2· sin(ϕ) ·sin(θ)], θ = 0 ..π, ϕ = 0 ..π, *coords* = *spherical*, *scaling* = *constrained*, *axes* = *boxed*);




```
> plot3d([sqrt(2*y-y^2-x^2),-sqrt(2*y-y^2-x^2),sqrt(y-y^2-x^2),-
sqrt(y-y^2-x^2)],x=-1..1,y=0..2,numpoints=2000);
```



By symmetry of the solid and of the density both moments $M_{yz}=M_{xy}=0$. *We have* **to** *calculate*

$$M_{xz}=\iiint y\,\rho\,dV=\int_0^\pi\int_0^\pi\int_{\sin\phi\sin\theta}^{2\sin\phi\sin\theta}(\rho\sin\phi\sin\theta)\,\rho\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta=\int_0^\pi\int_0^\pi\sin^2\phi\sin\theta$$

$$\left[\frac{1}{5}\rho^5\right]_{\sin\phi\sin\theta}^{2\sin\phi\sin\theta}d\phi\,d\theta=\frac{1}{5}\int_0^\pi\int_0^\pi\sin^2\phi\sin\theta[32-1]\sin^5\phi\sin^5\theta\,d\phi\,d\theta=$$

$$=\frac{31}{5}\int_0^\pi\sin^7\phi\,d\phi\int_0^\pi\sin^6\theta\,d\theta$$

We have $\sin^6\alpha=(1-\cos^2\alpha)^3=1-3\cos^2\alpha+3\cos^4\alpha-\cos^6\alpha$ so we have

$$\int_0^\pi\sin^7\phi\,d\phi=\int_0^\pi\sin\phi(1-3\cos^2\phi+3\cos^4\phi-\cos^6\phi)\,d\phi=\left[-\cos\phi+\cos^3\phi-\frac{3}{5}\cos^5\phi+\frac{1}{7}\cos^7\phi\right]_0^\pi=2$$

$$-2+\frac{6}{5}-\frac{2}{7}=\frac{32}{35}$$

and $\int_0^{\pi}\sin^6\theta\,d\theta=\int_0^{\pi}\left(1-3\cos^2\theta+3\cos^4\theta-\cos^6\theta\right)d\theta=$

$$\int_0^{\pi}\left(1-3\,\frac{1+\cos 2\theta}{2}+3\left(\frac{1+\cos 2\theta}{2}\right)^2-\left(\frac{1+\cos 2\theta}{2}\right)^3\right)d\theta=$$

$$=\int_0^{\pi}\left(1-\frac{3}{2}\left(1+\cos 2\theta\right)+\frac{3}{4}\left(1+2\cos 2\theta+\cos^2 2\theta\right)-\frac{1}{8}\left(1-3\cos 2\theta+3\cos^2 2\theta-\cos^3 2\theta\right)\right)d\theta=$$

Since $\int_0^{\pi}\cos 2\theta\,d\theta=\int_0^{\pi}\cos^3 2\theta\,d\theta=0$, *this is equal* **to**

$$=\int_0^{\pi}\left(1-\frac{3}{2}+\frac{3}{4}-\frac{1}{8}+\left(\frac{3}{4}-\frac{3}{8}\right)\left(\frac{1+\cos 4\theta}{2}\right)\right)d\theta=\ since\int_0^{\pi}\cos 4\theta\,d\theta=0=$$

$$=\pi\left(1-\frac{3}{2}+\frac{3}{4}-\frac{1}{8}+\frac{3}{16}\right)=\frac{5\pi}{16}$$

```
> 1-3/2+3/4-1/8+3/16;
```

$$\frac{5}{16} \qquad \textbf{(5)}$$

Thus, $M_{xz}=\frac{31}{5}\,\frac{32}{35}\,\frac{5\pi}{16}=\frac{62\pi}{35}$

The center of mass is $(\bar{x},\bar{y},\bar{z})=\left(\frac{M_{yz}}{m},\frac{M_{xz}}{m},\frac{M_{xy}}{m}\right)=\left(0,\frac{\frac{62\pi}{35}}{\frac{3\pi}{2}},0\right)=\left(0,\frac{124}{105},0\right).$