Concordia University

Department of Mathematics and Statistics

Course	**Number**		**Section**
STAT	250		A, B
Examination	**Date**	**Time**	**Pages**
Final	April, 2015	3 Hours	3
Instructors			**Marks**
Dr. Bellahnid, Dr. Zhou			60
Special Instructions:	**Closed Book Exam.** **1. Answer ALL questions.** **2. Calculators are allowed.** **3. Show your work in details and clearly identify.** **4. Tables of $N(0,1)$, F and χ^2 distributions are attached at the end of this booklet.**		

MARKS

[6] Q 1. Let X and Y be independent exponential random variables with mean 1. Find the density function for $Z = \frac{X}{Y}$.

A: Let $Z = X/Y, Y = Y$. Then $X = YZ, Y = Y, |J| = y$.

$$f(z, y) = f(zy, y)y = e^{-zy}e^{-y}y.$$

$$f(z) = \int_0^\infty e^{-zy}e^{-y}ydy = \int_0^\infty e^{-(z+1)y}ydy = \frac{1}{z+1}\int_0^\infty e^{-(z+1)y}dy = \frac{1}{(z+1)^2}.$$

Another solution:

$$P\{Z < z\} = P\{X < zY\} = \int_0^\infty dy \int_0^{yz} e^{-(x+y)}dx = 1 - \frac{1}{z+1}.$$

[8] Q 2. Suppose that you are a student worker in the department of mathematics and statistics and you agree to get paid randomly. Each week the chair flips a fair coin. Your pay for the week is \$80 if it comes up heads, and \$40 if it comes up tails. Your friend is working for another department and makes \$62 a week.

(a) Estimate the probability that your total earnings in 100 weeks are more than hers.

A:

$$V(\bar{X}) = \frac{1}{100}[\frac{1}{2}(80-60)^2 + \frac{1}{2}(40-60)^2] = 4.$$

$$P(\bar{X} > 62) = P\left(\frac{\bar{X}-60}{\sqrt{4}} > \frac{62-60}{2}\right) = P(Z > 1) = 0.1587.$$

(b) If you compare your earnings over 50 weeks instead of 100 weeks, would the above mentioned probability increase or decrease?

A: For $n = 50$, $V(\bar{X}) = 16$.

$$P(\bar{X} > 62) = P(Z > 2/\sqrt{8}).$$

[8] **Q 3.** $X_1, \dots, X_n$ is a sample of size n from a uniform population with density

$$f(x) = \begin{cases} 1/\theta, & 0 < x < \theta, \\ 0, & \text{elsewhere.} \end{cases}$$

We want to use $\hat{\theta} = X_{(n)}$ to estimate the parameter θ, where $X_{(n)}$ is the order statistics.

(a) Find the bias of $\hat{\theta}$.

A:

$$f_n(x) = nx^{n-1}/\theta^n.$$

$$EX_{(n)} = \frac{n}{\theta^n} \int_0^\theta x^n dx = \frac{n\theta}{n+1}.$$

(b) Find the mean square error of $\hat{\theta}$.

A:

$$E(X_{(n)} - \theta)^2 = EX_{(n)}^2 - 2\theta EX_{(n)} + \theta^2 = \frac{n\theta^2}{n+2} - \frac{2n\theta^2}{n+1} + \theta^2 = \frac{2\theta^2}{(n+1)(n+2)}.$$

[8] **Q 4.** After once again losing a football game to the university's rival, the alumni association conducted a survey to see if alumni were in favor of firing the coach. A random sample of 100 alumni from the population of all living alumni was taken. 64 of the alumni in the sample were in favor of firing the coach. Let p represent the proportion of all living alumni who favor firing the coach.

(a) Construct a 95% two-sided confidence interval for p.

A: $\left(0.64 - 1.96\sqrt{\frac{0.64\times0.36}{100}}, 0.64 + 1.96\sqrt{\frac{0.64\times0.36}{100}}\right) = (0.64 - 1.96 \times 0.048, 0.64 + 1.96 \times 0.048)$.

(b) For the 95% confidence coefficient, how large a sample size n would you need to estimate p with error at most 0.05?

A:

$$1.96\sqrt{\frac{0.64 \times 0.36}{n}} = 0.05.$$

$$n = (\frac{1.96 \times 0.48}{0.05})^2 = 354.$$

[8] **Q 5.** Suppose that $Y_1, Y_2, \ldots, Y_n$ denote a random sample from the gamma density function given by

$$f(y) = \begin{cases} \frac{y^{\alpha-1}e^{-y/\beta}}{\beta^\alpha\Gamma(\alpha)}, & 0 < y < \infty \\ 0, & \text{elsewhere.} \end{cases}$$

(a) find the sufficient statistics for α and β.

A:

$$L = \beta^{-n\alpha}\Gamma(\alpha)^{-n}(\prod_{i=1}^{n} y_i)^{\alpha-1}e^{-\frac{1}{\beta}\sum_{i=1}^{n} y_i}.$$

So, $(\prod_{i=1}^n Y_i, \sum_{i=1}^n Y_i)$ is jointly sufficient for (α, β).

(b) if α is known, find an MVUE for β.

A: $\bar{Y}$ is sufficient for β. Since $E\frac{1}{\alpha}\bar{Y} = \beta$, $\frac{1}{\alpha}\bar{Y}$ is a MVUE.

[8] **Q 6.** Suppose that $Y_1, Y_2, \ldots, Y_n$ constitute a random sample from the density function

$$f(y|\theta) = \begin{cases} e^{-(y-\theta)}, & y > \theta \\ 0, & \text{elsewhere.} \end{cases}$$

where θ is an unknown, positive constant.

(a) Find an estimator $\hat{\theta}_1$ for θ by the method of moments.

A:

$$EY = \int_\theta^\infty y e^{-(y-\theta)} dy = \int_0^\infty (x+\theta)e^{-x} dx = 1 + \theta.$$

$$1 + \theta = \bar{Y}.$$

$$\hat{\theta} = \bar{Y} - 1.$$

(b) Find an estimator $\hat{\theta}_2$ for θ by the method of maximum likelihood.

A:

$$L = e^{-\sum_{i=1}^n y_i + n\theta}.$$

$$\ln L = -\sum_{i=1}^n y_i + n\theta.$$

We should choose θ as large as possible to maximize L.

$$\hat{\theta} = \min\{y_i, i = 1, \dots, n\} = y_{(1)}.$$

[6] Q 7. The braking ability of two types of automobiles was compared. Random samples of 64 automobiles were tested for each type.The recorded measurement was the distance required to stop when the brakes were applied at 40 miles per hour. The computed sample means and variances were as follows:

$$\bar{y}_1 = 118, \quad \bar{y}_2 = 109, \quad s_1^2 = 102, \quad s_2^2 = 87.$$

Given confidence level 95%, do the data provide sufficient evidence to indicate a difference in the mean stopping distances of the two types of automobiles?

A: Large sample confidence interval. $EY_{1i} = \mu_1, \; EY_{2i} = \mu_2$. The confidence interval for $\mu_1 - \mu_2$ is

$$\left[\bar{y}_1 - \bar{y}_2 - \sqrt{\frac{s_1^2 + s_2^2}{64}}, \; \bar{y}_1 - \bar{y}_2 + \sqrt{\frac{s_1^2 + s_2^2}{64}}\right] = [16 + 1.72, 16 - 1.72].$$

There is significant evidence for a difference since $0 \notin [16 + 1.72, 16 - 1.72]$.

[8] **Q 8.** A pharmaceutical manufacturer purchases a particular material from two different suppliers. The mean level of impurities in the raw material is approximately the same for both suppliers, but the manufacturer is concerned about the variability of the impurities from shipment to shipment. If the level of impurities tends to vary excessively for one source of supply, it could affect the quality of the pharmaceutical product. To compare the variation in percentage impurities for the two suppliers, the manufacturer selects ten shipments from each of the two suppliers and measures the percentage of impurities in the raw material for each shipment. The sample means and variances are shown in the table:

SupplierA	*SupplierB*
$\bar{y}_1 = 1.89$	$\bar{y}_2 = 1.85$
$s_1^2 = 0.273$	$s_2^2 = 0.094$
$n_1 = 10$	$n_2 = 10$

Assume the level of impurity roughly follows a normal distribution for each supplier.

(a) Do the data provide sufficient evidence to indicate a difference in the variability of the shipment impurity levels for the two suppliers? Test using $\alpha = 0.10$. Based on the result of your test, what recommendation would you make to the pharmaceutical manufacturer?

A: Small sample test of $\mu_1 - \mu_2$.

$H_0 : \sigma_1^2 = \sigma_2^2,\ \ H_a : \sigma_1^2 \neq \sigma_2^2$.

Test statistics $F = S_1^2/S_2^2$. $F_{0.05}(9, 9) = 3.18,\ F_{0.95}(9, 9) = 0.31$.

Reject region $(-\infty, 0.31] \cup [3.18, \infty)$.

The observed F value is $F = \frac{0.273}{0.094} = 2.90$. So, we accept H_0 at significance level 0.1.

(b) Find a 0.05 confidence interval for σ_2^2 and interpret your results.

A: Confidence interval for σ_2^2:

$$\left(\frac{9s_2^2}{\chi^2_{0.025}(9)}, \frac{9s_2^2}{\chi^2_{0.975}(9)}\right) = \left(\frac{9 \times 0.094}{19.02}, \frac{9 \times 0.094}{2.7}\right).$$