2125 W15 final · Sol^n + Mk'ing scheme

1. Let $A \subseteq \mathbf{R}$. We say that a real number $M \in \mathbf{R}$ is a *maximum of* A, if M is an upper bound for A and $M \in A$.

1 a) State necessary and sufficient conditions for the supremum of A, sup A to exist.

1 b) Given an example of a set A where sup A exists but max A does not. (You do not need to prove that your example is a good one. Just be sure it is.)

2 c) Prove that if max A exists, then so does sup A and sup $A = \max A$.

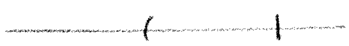
2 d) Now suppose $A \subset \mathbf{Z}$ is a non-empty subset of the integers which is bounded above. Prove that sup A is an integer.

a) A must be non-empty and bounded above

b) Let $A = (0,1) \subset \mathbb{R}$. Then $\sup A = 1$, but Max A does not exist.

c) Let $m = \max A$. Then $\forall a \in A$, $a \le m$ $\frac{1}{2}$, and $m \in A$.

Since m is an upper bound for A, it suffices to show the 2nd property of sup A. So let $\varepsilon > 0$.

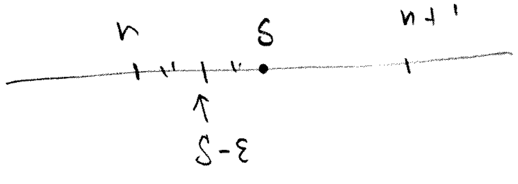


Then $m \in (m-\varepsilon, m]$ and $m \in A$, so m satisfies both properties of sup A. Hence $m = \sup A$

d) Let $s = \sup A$ (which we know exists by (a)). $\frac{1}{2}$ We claim $s \in \mathbb{Z}$.

If not, then $\exists n \in \mathbb{Z}$ s.t. $s \in (n, n+1)$.

Let $\varepsilon = \frac{s-n}{2} > 0$. Then there exists $a \in \left(\frac{n+s}{2}, s\right] \cap A$. But



a is also an integer, and there are no integers in $\left(\frac{n+s}{2}, s\right] \subset (n, n+1)$.

Hence $s \in \mathbb{Z}$.

(max 1 if assume max A exists!)

2. Let $\{a_n\}_{n\geq 1}$ be a real sequence.

1 a) Define what is meant by "$\{a_n\}_{n\geq 1}$ is a Cauchy sequence."

1 b) Give a statement, different from the definition, which is equivalent to "$\{a_n\}_{n\geq 1}$ is a Cauchy sequence."

1 c) Define what is meant by "b is an accumulation point of $\{a_n\}_{n\geq 1}$."

1 d) State the Bolzano-Weierstrass theorem for real sequences.

2 e) Suppose $\{a_n\}_{n\geq 1} \subset [a,b]$ for some real numbers $a < b$. Prove that the sequence

$$\left\{\frac{a_n}{n}\right\}_{n\geq 1}$$

is Cauchy.

a) $\{a_n\}_{n\geq1}$ is Cauchy if $\forall\, \varepsilon > 0\ \exists N \in \mathbb{N}$ st $\forall\, m,n \geq N$, $|a_n - a_m| < \varepsilon$

b) $\{a_n\}_{n\geq1}$ is Cauchy iff $\{a_m\}_{n\geq1}$ converges

c) The number b is an accumulatoin point of $\{a_n\}_{n\geq1}$ if there is a subsequence of $\{a_n\}_{n\geq1}$ which converges to b

d) B-W: Every bounded real sequence has an accumulation point

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e) Let $M = \max\{|a|,|b|\}$. Then, $\forall\, x \in [a,b]$, $|x| \leq M$ $\frac{1}{2}$

Hence, $\forall\, n \geq 1$, $0 \leq \left|\frac{a_m}{n}\right| \leq \frac{M}{n}$. Since $\lim_{n\to\infty} \frac{M}{n} = 0$, by the Squeeze thm, $\lim_{n\to\infty} \frac{a_m}{n} = 0$. Hence $\left\{\frac{a_n}{n}\right\}_{n\geq1}$ is convergent and so Cauchy. ↑ 1

3. a) Let $\{b_n\}_{n\geq 1}$ be a real sequence. Define

$$\text{“The series } \sum_{n=1}^{\infty} b_n \text{ is absolutely convergent.”}$$

1 b) Prove by induction for that $2^n \geq 2^{n-1} + n$, $\forall n \in \mathbf{N}, n \geq 1$.

2 c) Prove that the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n - n}$ is absolutely convergent.

1 d) Give an example of a convergent series which is not absolutely convergent. (no pf req'd)

1 e) Give an example of an absolutely convergent series which is not convergent, or give reasons it is impossible to do so.

(In (c)– (e), you may use known tests and theorems, but be sure to verify their hypotheses.)

a) The series $\sum_{n=1}^{\infty} b_n$ is absolutely convergent if $\sum_{n\geq 1} |b_n|$ converges.

b) The statement for $n=1$ is $2 \geq 1+1 = 2$, which is true

Now suppose $2^n \geq 2^{n-1} + n$. Then $2^{n+1} \geq 2(2^{n-1}+n) = 2^n + 2n$

$\geq 2^n + n + 1$ (as $n \geq 1$)

Hence, $\forall n \geq 1$, $2^n \geq 2^{n-1} + n$. ⑤38

c) By (b), $2^n - n \geq 2^{n-1} \geq 0 \; \forall n \geq 1$, so $\left|\frac{(-1)^n}{2^n - n}\right| \leq \frac{1}{2^{n-1}}$.

But $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$ converges (it is a geometric series with ratio $\frac{1}{2} < 1$)

Hence, by comparison, $\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n - n}$ converges absolutely.

d) $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ converges (by Leibniz) but $\sum_{n\geq 1} \frac{1}{n}$ diverges.

e) It is impossible. Every absolutely convergent series is convergent

4. Let $A \subseteq \mathbf{R}$, $a \in A$ and $f : A \to \mathbf{R}$.

1 a) Define

" The function f is continuous at a."

1 b) Prove that $\forall x \in \mathbf{R}, \quad |x-1| < 1 \Longrightarrow 1 + x^2 > 1.$

1 c) Prove that $\forall x \in \mathbf{R}, \quad |x-1| < 1 \Longrightarrow |1+x| < 3.$

2 d) Define $f : \mathbf{R} \to \mathbf{R}$ by $f(x) = \dfrac{1}{1+x^2}$.

Prove, using the definition (i.e., using "$\varepsilon - \delta$, etc.") that f is continuous at 1.

1 e) Give an example of a function $g : (0,1] \to \mathbf{R}$ which is continuous on $(0,1]$ but is not *uniformly continous* on $(0,1]$.

a) The function f is cts at a if $\forall \varepsilon > 0\ \exists \delta$ st $\forall x \in B(a,\delta) \cap A$, $f(x) \in B(a,\varepsilon)$..

b) Note that $|x-1| < 1 \iff 0 < x < 2 \Rightarrow x > 0 \Rightarrow 1+x^2 > 1$.

c) Note that $|x-1| < 1 \iff 0 < x < 2 \iff 1 < 1+x < 3$
$\Rightarrow |1+x| = 1+x < 3$.

d) Note that $|f(x) - f(1)| = \left| \dfrac{1}{1+x^2} - \dfrac{1}{2} \right| = \left| \dfrac{2-(1+x^2)}{21+x^2} \right|$

$= \dfrac{|1-x^2|}{2(1+x^2)} = \dfrac{|1+x||1-x|}{2(1+x^2)} \cdot \dfrac{1}{2}$ Let $\varepsilon > 0$ and choose
$\delta = \min\left(1, \frac{2\varepsilon}{3}\right)$. Then $|x-1| < \delta \Rightarrow$ (by (b) & (c)) that

$|f(x) - 1| < \dfrac{3}{2} \cdot \dfrac{|1-x|}{1} \leq \dfrac{3}{2} \cdot \dfrac{2\varepsilon}{3} = \varepsilon$. Hence f is cts at 1.

e) Let $g(x) = \frac{1}{x}$. Then as we saw, g is not unif. cts on $(0,1]$
(If it were, it would have a cts extension to $[0,1]$, which is impossible.

5. Let A and C be subsets of $\mathbf{R}^p$.

1 a) Define "A is an open subset of $\mathbf{R}^p$."

1 b) Prove from the definition that if A and C are open, then so is their intersection $A \cap C$.

1 c) Give two different characterizations of "C is compact", one of which is in terms of sequences in C.

1 d) Suppose $A \neq \emptyset, a \in A$ and that $X = \{r \mid r \in \mathbf{R},\ r > 0, \text{ and } \mathbf{B}(a,r) \subset A\}$ is unbounded. Prove that $A = \mathbf{R}^p$.

2 e) Now suppose $p = 1$, and that $A \subseteq \mathbf{R}$ is both open and closed. Prove carefully that $A = \emptyset$ or $A = \mathbf{R}$.

a) A is open if $\forall a \in A\ \exists r > 0$ st. $B(a,r) \subseteq A$, where
$B(a,r) = \{v \in \mathbb{R}^p \mid \|v - a\| < r\}$.

b) If $A \cap C = \phi$, as ϕ is open, so is $A \cap C$. Let $v \in A \cap C$. Then $\exists\, r_A > 0$ and $r_C > 0$ s.t. $B(v, r_A) \subseteq A$ and $B(v, r_B) \subseteq C$. Let $r = \min(r_A, r_C)$. Then $B(v,r) \subseteq B(v, r_A) \cap B(v, r_B) \subseteq A \cap C$. Hence $A \cap C$ is open

c) C is compact iff C is closed and bounded
C is compact iff every sequence in C has an accumulation pt in C
C is compact iff every open cover of C has a finite subcover

d) Let $v \in \mathbb{R}^p$, and set $s = \|v - a\|$. Since X is unbounded, $\exists\, r \in X$ with $r > s$. Hence $v \in B(a,s) \subseteq A$. Thus $A = \mathbb{R}^p$

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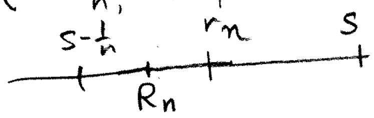
e) Suppose $A \neq \phi$ and $a \in A$. Set $X = \{r > 0 \mid (a-r, a+r) \subseteq A\}$.
Since A is open, X is non-empty. If X is bounded, $s = \sup X$ exists. Now since $s = \sup X$, $\forall n \geq 1$, $\exists\, r_n \in X \cap (s - \frac{1}{n}, s]$.



So $\forall n \geq 1$, $(a - r_n, a + r_n) \subseteq A$. In particular,
Since $R_n = \frac{s - \frac{1}{n} + r_n}{2} < r_n$, both $a - R_n$ and $a + R_n \in A$. But $R_n \to s$ so
$a - R_n \to a - s$ and $a + R_n \to a + s$. As A is closed, $a \pm s \in A$.
Since A is open, $\exists\, \delta > 0$ s.t. $(a - s - \delta, a - s + \delta) \subseteq A$ and $(a + s - \delta, a + s + \delta)$
$\subseteq A$. Hence $s + \delta \in X$, a contradiction. Hence X is unbdd and
so by (d), $A = \mathbb{R}$.

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6. Let $f, g, h : [-1,1] \to \mathbf{R}$ be three functions.

1 a) If $c \in (-1,1)$, define "f is differentiable at c."

1½ b) If g is continuous at 0, and $f(x) = xg(x), \forall x \in [-1,1]$, prove that f is differentiable at 0. (We do *not* assume that g is differentiable anywhere.)

1½ c) State the the Mean Value Theorem *for derivatives.*

2 d) Now suppose h is continuous at 0, and is differentiable on $(-1,0)$ and $(0,1)$. If $\lim_{x\to 0} h'(x)$ exists, prove that h *is also* differentiable at 0 and that $\lim_{x\to 0} h'(x) = h'(0)$. (Hint: Use the MVT.)

a) f is diff'ble at c if $\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$ exists.

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b) Consider $x \neq 0$ and the expression $\frac{f(x) - f(0)}{x - 0} = \frac{xg(x)}{x} = g(x)$.

1½ Since g is cts at 0, $\lim_{x\to 0} g(x) = g(0)$ exists, $f'(0)$ $(= g(0))$ exists

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c) If $f : [a,b] \to \mathbb{R}$ $(a<b)$ is cts on $[a,b]$ ½ and diff'ble on (a,b), ½

1½ then $\exists\, \xi \in (a,b)$ s.t. $\frac{f(b) - f(a)}{b-a} = f'(\xi)$ ½

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2 d) We know h is cts on $[-\frac{1}{2}, 0]$, and $[0, \frac{1}{2}]$ and differentiable on $(-\frac{1}{2}, 0)$ & $(0, \frac{1}{2})$. Suppose $x \neq 0$. Then by the MVT

$\frac{h(x) - h(0)}{x - 0} = h'(\xi(x))$ for some $\xi(x) \in (x, 0)$ or $(0, x)$, depending on the sign of x. Hence $|\xi(x)| < |x|$. Since $\lim_{x\to 0} \xi(x) = 0$ and $\lim_{y \to 0} h'(y)$ exists, by composition of limits, $\lim_{x\to 0} h'(\xi(x))$ exists.

Hence $h'(0)$ exists and is indeed $\lim_{x \to 0} h'(x)$.

7. Let A be a subset of $\mathbf{R}$, and $f_n : \mathbf{R} \to \mathbf{R}$ a sequence of functions . Now consider the series $\sum_{n=0}^{\infty} f_n(x)$.

1 a) If $f : A \to \mathbf{R}$ is a function, define "$\sum_{n=0}^{\infty} f_n$ converges uniformly on A to f."

2 b) State the Weierstrass M-test.

1 c) Prove that $\forall K \in \mathbf{R}$, the series $\sum_{n=0}^{\infty} \frac{K^{2n+1}}{(n+1)!}$ converges.

Now define $f : \mathbf{R} \to \mathbf{R}$ by

$$f(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(n+1)!}.$$

1 d) Explain briefly (use a theorem!) why f is differentiable on $\mathbf{R}$, and give its derivative.

1 e) Find $\int_0^a f$, carefully justifying your steps.

a) $\sum_{n=0}^{\infty} f_n$ converges uniformly to f on A if $\forall \varepsilon > 0$ $\exists N$ st $\forall a \in A$, $\forall n \geq N$, $\left| \sum_{k=1}^{n} f(a) - f(a) \right| < \varepsilon$

b) If $\{M_n\}_{n \geq 1}$ is a positive sequence st 1) $\forall x \in A$, $|f_n(x)| \leq M_n$ &
2) $\sum_{n=0}^{\infty} M_n$ converges,

then $\sum_{n=0}^{\infty} f_n$ converges uniformly on A

c) Let $M = |K|$. Then $\frac{M^{2n+3}}{(n+2)!} \cdot \frac{(n+1)!}{M^{2n+1}} = \frac{M^2}{n+2} \to 0$ for all M.

Hence by the limit ratio test, $\sum_{n=0}^{\infty} \frac{K^{2n+1}}{(n+1)!}$ converges (absolutely), $\forall K$.

d) By (c), $\sum \frac{x_0^{2n+1}}{(n+1)!}$ converges, $\forall x_0 \in \mathbb{R}$. So by a theorem, (on power series) f is differentiable on $(-\rho, \rho)$ $\forall \rho < |x_0|$, $\forall x_0 \in \mathbb{R}$. Hence f is defined on all of $\mathbb{R}$. The same theorem guarantees that we can differentiate term by term, so $f'(x) = \sum_{n=0}^{\infty} \frac{(2n+1)x^{2n}}{(n+1)!}$

7(e) By the W-M test, and part (c), the series defining f converges uniformly on $[-K,K]$ for any $K \geq 0$.

Choose K st $K \geq |a|$.

Hence we may integrate term by term, and obtain

$$\int_0^a f = \int_0^a \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(n+1)!} = \sum_{n=0}^{\infty} \int_0^a \frac{x^{2n+1}}{(n+1)!}$$

$$= \sum_{n=0}^{\infty} \frac{a^{2n+2}}{(2n+2)(n+1)!}$$

8. Define $f : [0,1] \to \mathbf{R}$ by

$$f(x) = \begin{cases} 2 & \text{for } x \in [0, \frac{1}{2}) \\ 0 & \text{for } x = \frac{1}{2} \\ -1 & \text{for } x \in (\frac{1}{2}, 1] \end{cases}$$

1 a) State the Intermediate Value Theorem.

1 b) Prove that f is not continuous on $[0,1]$. You may use a theorem.

1 c) Carefully state a necessary and sufficient condition for f to be Riemann (-Darboux) integrable in terms of upper sums $U(f,P)$ and lowers sums $L(f,P)$, where P denotes a partition of $[0,1]$. (You may give the definition or an equivalent condition.)

2 d) For $n \in \mathbf{N}, n \geq 3$, let P_n be the partition $P_n = \{0, \frac{1}{2} - \frac{1}{n}, \frac{1}{2} + \frac{1}{n}, 1\}$. Find $U(f, P_n)$ and $L(f, P_n)$, for all $n \geq 3$.

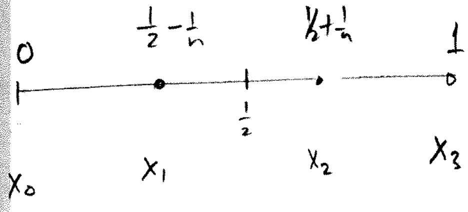
1 e) Use your result in (d) and your response in (c) to prove that f is integrable, and find $\int_0^1 f$.

a) If $f : [a,b] \to \mathbb{R}$ is cts on $[a,b]$ and l is between $f(a)$ & $f(b)$, $\exists\, \xi \in [a,b]$ s.t. $f(\xi) = l$.

b) Note that $f(0) = 2$, $f(1) = -1$ and $1 \in [-1, 2]$. However there is no $\xi \in [0,1]$ s.t. $f(\xi) = 1$. Hence f is not cts on $[0,1]$.

c) f is Riemann-Darboux integrable on $[0,1]$ iff $\forall\, \varepsilon > 0$ $\exists$ partition P of $[0,1]$ s.t. $U(f,P) - L(f,P) < \varepsilon$.

d) Note: $m_1(f) = 2 = M_1(f)$, $m_2(f) = -1$, $M_2(f) = 2$ and $m_3(f) = -1 = M_3(f)$. (for all $n \geq 3$)



Hence $L(f,P_n) = 2\left(\frac{1}{2} - \frac{1}{n}\right) - 1\left(\frac{2}{n}\right) - 1\left(\frac{1}{2} - \frac{1}{n}\right)$

$= 1 - \frac{2}{n} - \frac{2}{n} - \frac{1}{2} + \frac{1}{n} = \underline{\frac{1}{2} - \frac{3}{n}}$

And $U(f,P_n) = 2\left(\frac{1}{2} - \frac{1}{n}\right) + 2\left(\frac{2}{n}\right) - 1\left(\frac{1}{2} - \frac{1}{n}\right) = 1 - \frac{2}{n} + \frac{4}{n} - \frac{1}{2} + \frac{1}{n} = \underline{\frac{1}{2} + \frac{3}{n}}.$

e) Since $U(f,P_n) - L(f,P_n) = \frac{6}{n}$, choose $n > \frac{6}{\varepsilon}$ so that $U(f,P_n) - L(f,P_n) = \frac{6}{n} < \varepsilon$.
Hence f is int. on $[0,1]$ and $\int_0^1 f = \frac{1}{2}$

9. Define a function $f : \mathbf{R} \to \mathbf{R}$ by by $f(x) = \int_0^x \frac{1}{1+4t^2}\, dt$.

1 a) *Briefly* explain why f is differentiable on $\mathbf{R}$. (Use theorems!)

$1\frac{1}{2}$ b) State the Mean Value Theorem *for integrals.*

1 c) Prove that if we define $g : \mathbf{R} \to \mathbf{R}$ by $g(x) = 2f(\frac{x}{2})$, then g is strictly increasing on $\mathbf{R}$.

1 d) Find $g'(x)$.

Now denote $g(1) = p$. Let

$$h : [0, p] \to [0, 1]$$

be the inverse function for g, which we know exists by parts (c) and (d), and define $k : [0, p] \to \mathbf{R}$ by

$$k(x) = \sqrt{1 + h^2(x)}, \quad \forall x \in [0, p].$$

$1\frac{1}{2}$ e ~~d)~~ *Briefly* explain why h and k are differentiable on $(0, p)$, and show that $\frac{1}{2}$

$$h'(x) = k^2(x), \quad 1/2$$

and

$$k'(x) = h(x)k(x), \quad 1/2$$

for all $x \in (0, p)$. (Use theorems!)

(1π)

a) The integrand defing f is cts, and by the FTC, $f'(x) = \frac{1}{1+4x^2}$.

b) If g is cts on $[a, b]$, then $\exists \xi \in [a, b]$ s.t. $\int_a^b g = g(\xi)(b-a)$

c) The fn g will be strictly increasing iff f is : But if $x > y$ then $f(x) - f(y) = \int_y^x \frac{1}{1+4t^2}\, dt = \left(\frac{1}{1+4\xi^2}\right) \cdot (x-y)$ for some $\xi \in [y, x]$.

Hence $f(x) > f(y)$, so $g(x) > g(y)$

d) By the Chain rule, $g'(x) = \frac{2}{2} f'(\frac{x}{2}) = \frac{1}{1+x^2}$.

e) The fn h is differentiable on $(0, p)$ because $h = g^{-1}$ and $g' \neq 0$ on $[0, 1]$. The fn k is diffrentiable on $(0, p)$ because . . .

9 (d) (cont)... $1+h^2 > 1 > 0$ on $(0,p)$, and $y \mapsto \sqrt{y}$ is diff if $y > 0$

We know $h'(x) = \dfrac{1}{g'(h(x))} = \left(\dfrac{1}{1+h^2(x)}\right)^{-1} = 1+h^2(x)$
$(h = g^{-1})$ $\qquad = k^2(x)$

Moreover, $k'(x) = \dfrac{\frac{1}{2} \cdot 2h'(x)h(x)}{\sqrt{1+h^2(x)}} = \dfrac{k^2(x)h(x)}{k(x)} = k(x)h(x).$

10. (Bonus) a) Suppose $\{F_n \mid n \in \mathbf{N}, n \geq 1\}$ is a sequence of closed subsets of $\mathbf{R}^p$, such that

(i) F_1 is bounded, and

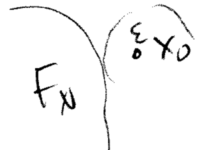
(ii) $\bigcap_{i=1}^{n} F_i \neq \emptyset$, for all $n \in \mathbf{N}$.

Prove that $\bigcap_{i=1}^{\infty} F_i \neq \emptyset$.

~~b) Prove that if A is non-empty, open and closed, then $A = \mathbf{R}$.~~

Since F_1 is closed and bdd, it is compact. Note that $\forall n$, $\bigcap_{i=1}^{n} F_i \subseteq F_1$. Choose $x_n \in \bigcap_{i=1}^{n} F_i$. Then $\{x_n\}_{n \geq 1}$ is a sequence in the compact set F_1, so it has a subsequence, say $\{x_{n_k}\}_{k \geq 1}$ which converges to (say) x_0.

Claim: $x_0 \in \bigcap_{i=1}^{\infty} F_i$. Suppose not. Then $\exists N$ st. $x_0 \notin F_N$. As F_N^c is open, $\exists \varepsilon > 0$ st $B(x_0, \varepsilon) \subset F_N^c$.



Since $\lim_{k \to \infty} x_{n_k} = x_0$, $\exists M$ st $\forall k \geq M$, $x_{n_k} \in B(x_0, \varepsilon)$. In particular, there will be k st $n_k \geq N$ so $x_{n_k} \in B(x_0, \varepsilon) \subset F_N^c$.

But $x_{n_k} \in \bigcap_{i=1}^{n_k} F_i \subseteq F_N$ (as $n_k \geq N$). This is a contradiction. Hence $x_0 \in \bigcap_{i=1}^{\infty} F_i$ and so the latter is non-empty