Faculty of Science »
Department of mathematics and statistics

MAT 1341A Fall 2015 – Final Exam

17-December, 2015.

Instructor - Barry Jessup

Family Name:____________________

First Name:____________________ Seat number: ________

Student number:____________________

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Some Advice

Take a few minutes to read the entire paper before you begin to write, and read each question carefully. The multiple choice questions are only worth 1 point and questions 11-15 are worth 6 points each. Make a note of the questions you feel confident you can do, and try those first: you do not have to do the questions in the order they are presented.

Instructions

1. You have 3 hours to complete this exam.
2. **This is a closed book exam, and no notes of any kind are permitted. The use or possession at your exam desk of calculators, cell phones, or similar devices is not permitted. By signing the attendance sheet you acknowledge that you will comply with these conditions.**
3. Questions 1 to 10 are multiple choice. These questions are worth **1 point** each and no part marks will be given. Please record your answers in the boxes to the right of the numbers 1-10 in the table here.
4. Questions 11 – 15 require a complete solution, and are worth 6 points each.
   Question 16 is a bonus question worth 4 points and should only be attempted after all other questions have been completed and checked, since bonus marks are much harder to earn.
   Spend your time accordingly.
   Answer questions 11 – 16 in the space provided, and use the backs of pages if necessary.
5. **The correct answer requires justification written legibly and logically: you must convince me that you know why your solution is correct.**
6. Where it is possible to check your work, do so.

Good luck! Bonne chance!

1. For a **homogeneous** system of 1500 equations in 2015 unknowns, answer the following three questions:

- Can the system be inconsistent?
- Can the system have infinitely many solutions?
- Can the system have exactly one solution?

A. No, Yes, No.
B. Yes, Yes, Yes.
C. Yes, Yes, No.
D. No, No, No.
E. Yes, No, Yes.
F. No, No, Yes.

$$1500\updownarrow \underbrace{\left[\begin{array}{c|c} \quad A \quad & \begin{matrix}0\\ \vdots \\ 0\end{matrix}\end{array}\right]}_{\leftarrow 2015 \rightarrow}$$

Homogeneous systems are always consistent because $x_1 = x_2 = \dots = x_n = 0$ is always a solution.

It can have infinitely many solutions, e.g.:

$$\left[\begin{array}{ccc|c} 0 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{array}\right]$$

It can't have exactly one solution because the general solution has at least 515 free parameters.

2. Suppose $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$. Which one of the following statements is true ?

A. A^{-1} does not exist.
B. The third row vector of A^{-1} is $(-1, 1, 1)$.
C. The second row vector of A^{-1} is $(0, 1, 1)$.
D. The first row vector of A^{-1} is $(1, -1, 2)$.
E. The second column vector of A^{-1} is $(0, 2, -1)$.
F. The first column vector of A^{-1} is $(1, 2, -1)$.

Compute the inverse:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 2 & -1 & -1 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

A^{-1}

So, A^{-1} exists and its third row vector is $(-1, 1, 1)$!

3. Let A be a 13×8 matrix such that $\operatorname{rank} A = 6$. As usual, $\ker A = \{x \in \mathbf{R}^8 \mid Ax = 0\}$ and $\text{col } A = \{Ax \mid x \in \mathbf{R}^8\}$. Which of the following statements is true?

A. $\dim \text{col } A = 6$, and $\dim \ker A = 1$

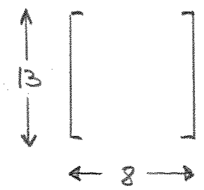
B. $\text{col } A = \mathbf{R}^{13}$, and $\dim \ker A = 6$

C. $\text{col } A = \mathbf{R}^8$, and $\dim \ker A = 5$

D. $\dim \text{col } A = 6$, and $\dim \ker A = 2$

E. $\dim \text{col } A = 6$, and $\dim \ker A = 7$

F. $\text{col } A = \{0\}$, and $\dim \ker A = 7$



$\text{rank}(A) = 6$

$\Rightarrow \dim \text{col}(A) = 6$

Recall that:
$\text{rank}(A) = \dim \text{Col}(A) = \dim \text{Row}(A)$

$\dim \ker(A) = \#\text{ columns of } A - \text{rank}(A) = 8 - 6 = 2$

Rank-Nullity-Theorem

4. Let $X = \{(x, y, z) \in \mathbf{R}^3 \mid x + y = 0\}$. Which one of the following statements is true?

A. X is a subspace of $\mathbf{R}^3$ and $\dim X = 3$.

B. X is a subspace of $\mathbf{R}^2$ and $\dim X = 1$.

C. X is a plane in $\mathbf{R}^3$ through the origin which is parallel to the z–axis.

D. X is a not a subspace of $\mathbf{R}^3$.

E. X is a line through the origin in $\mathbf{R}^3$.

F. X is a plane in $\mathbf{R}^3$ through the origin which is parallel to the x–axis.

$$X = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; x = -y \right\} = \left\{ \begin{pmatrix} -y \\ y \\ z \end{pmatrix} \;\middle|\; y, z \in \mathbb{R} \right\}$$

$$= \left\{ y \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + z \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \;\middle|\; y, z \in \mathbb{R} \right\} = \text{span} \underbrace{\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}}_{\text{linearly independent}}$$

So, X is a plane in $\mathbb{R}^3$ through the origin; it contains the z-axis and is therefore also parallel to the z-axis.

5. Recall that A^t denotes the transpose of the matrix A. The dimension of

$$S = \{A \in \mathbf{M}_{33}(\mathbf{R}) \mid A = -A^t\}$$

is:

A. 1
B. 2
C. 3
D. 4
E. 6
F. 9

Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$. Then,

$$A = -A^T \Leftrightarrow \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} -a & -d & -g \\ -b & -e & -h \\ -c & -f & -i \end{bmatrix} \Leftrightarrow \begin{matrix} a=-a & f=-h \\ b=-d & g=-c \\ c=-g & h=-f \\ d=-b & i=-i \\ e=-e & \end{matrix}$$

$$\Leftrightarrow \quad a=e=i=0 \quad \underline{\text{and}} \quad b=-d,\ c=-g,\ f=-h$$

So, $S = \left\{ \begin{bmatrix} 0 & -d & -g \\ d & 0 & -h \\ g & h & 0 \end{bmatrix} \,\middle|\, d,g,h \in \mathbb{R} \right\}$

$$= \left\{ d \cdot \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + g \cdot \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + h \cdot \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \,\middle|\, d,g,h \in \mathbb{R} \right\}$$

$$= \text{Span} \underbrace{\left\{ \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \right\}}_{\text{linearly independent}}$$

6. Which two of the following statements are true?

I. $\{2\sin^2 x, 3\cos^2 x, 2\}$ is linearly independent in $\mathbf{F}(\mathbf{R}) = \{f \mid f : \mathbf{R} \to \mathbf{R}\}$.
II. $\mathbf{R}^2$ has a spanning set with 3 vectors.
III. The set of all solutions to any homogeneous linear system is a subspace.
IV. If u and v are linearly independent vectors in $\mathbf{R}^2$, then $\{u, v\} = \text{span}\{u, v\}$.

A. *I* and *II*.
B. *I* and *III*.
C. *I* and *IV*.
D. *II* and *III*.
E. *II* and *IV*.
F. *III* and *IV*.

I Recall that $\sin^2(x) + \cos^2(x) = 1$. Therefore,

$$3 \cdot (2\sin^2(x)) + 2 \cdot (3\cos^2(x)) - 3 \cdot 2$$
$$= 6\sin^2(x) + 6\cos^2(x) - 6$$
$$= 6 \cdot (\sin^2(x) + \cos^2(x) - 1)$$
$$= 6 \cdot 0 = 0$$

So, $\{2\sin^2(x), 3\cos^2(x), 2\}$ is linearly dependent.

II Yes, $\{\binom{1}{0}, \binom{0}{1}, \binom{1}{1}\}$ spans $\mathbb{R}^2$.

III Yes, if $Ax = 0$ denotes the homogeneous system, then the general solution is equal to $\ker(A)$, which is a subspace.

IV No, consider $\binom{1}{0}$ and $\binom{0}{1}$. Then $\{\binom{1}{0}, \binom{0}{1}\}$ consists of two vectors only whereas $\text{Span}\{\binom{1}{0}, \binom{0}{1}\}$ consists of infinitely many vectors.

7. If $C = \begin{bmatrix} 0 & 3 & 1 \\ 1 & 2 & 1 \end{bmatrix}$ and D is a $3 \times n$ matrix , then the first row of the matrix CD is always

A. undefined unless $n = 2$.

B. twice the first row of D.

C. the same as the first row of D.

D. the same as the second row of D.

E. the sum of the first and the second rows of D.

F. the sum of three times the second row of D and the third row of D.

$$\underbrace{\begin{bmatrix} 0 & 3 & 1 \\ 1 & 2 & 1 \end{bmatrix}}_{C} \cdot \underbrace{\begin{bmatrix} \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \vdots \end{bmatrix}}_{D} = \cdots$$

Since the first row of C is [0 3 1], the first row of CD is the sum of three times the second row and the third row of D.

8. If $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 2$, find

$$\begin{vmatrix} b+4c & e+4f & h+4i \\ -3c & -3f & -3i \\ 5a & 5d & 5g \end{vmatrix}$$

A. -120
B. -30
C. -24
D. 24
E. 30
F. 120

$$\det \begin{bmatrix} b+4c & e+4f & h+4i \\ -3c & -3f & -3i \\ 5a & 5d & 5g \end{bmatrix} = (-3) \cdot 5 \cdot \det \begin{bmatrix} b+4c & e+4f & h+4i \\ c & f & i \\ a & d & g \end{bmatrix}$$

$$= (-3) \cdot 5 \cdot \det \begin{bmatrix} b & e & h \\ c & f & i \\ a & d & g \end{bmatrix}$$

$$= (-3) \cdot 5 \cdot \det \begin{bmatrix} b & c & a \\ e & f & d \\ h & i & g \end{bmatrix}$$

transpose: $\det(A^T) = \det(A)$

Switch two columns!

$$= (-3) \cdot 5 \cdot \left(-\det \begin{bmatrix} a & c & b \\ d & f & e \\ g & i & h \end{bmatrix} \right)$$

$$= (-3) \cdot 5 \cdot \det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$= (-3) \cdot 5 \cdot 2 = -30$$

9. The vectors $u_1 = (1,1,2), u_2 = (1,-1,0)$, and $u_3 = (1,1,-1)$ form an orthogonal basis of $\mathbf{R}^3$. If we write $(1,-1,1) = a_1u_1 + a_2u_2 + a_3u_3$, what is a_2?

A. -1
B. 1
C. $-\frac{\sqrt{2}}{2}$
D. $\frac{\sqrt{2}}{2}$
E. $-\frac{1}{3}$
F. $\frac{1}{3}$

$$a_2 = \frac{u_2 \cdot \begin{pmatrix}1\\-1\\1\end{pmatrix}}{u_2 \cdot u_2} = \frac{\begin{pmatrix}1\\-1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\1\end{pmatrix}}{\begin{pmatrix}1\\-1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\0\end{pmatrix}} = \frac{2}{2} = 1$$

10. Let A be a 3 by 3 matrix with real entries. Which of the following statements is **equivalent to**

"A is not diagonalizable over the reals."

A. Not all of the eigenvalues of A are real.

B. A does not have any real eigenvectors.

C. A does not have three distinct real eigenvalues.

D. A does not have three independent eigenvectors in $\mathbf{R}^3$.

E. A is upper triangular.

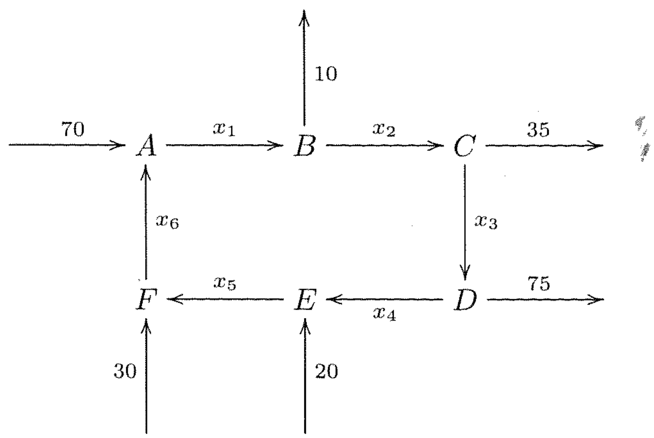
F. A is not invertible.

By definition, A is diagonizable over the reals if there is a basis of $\mathbb{R}^3$ consisting entirely of eigenvectors of A. In other words, if there are three (linearly) independent eigenvectors of A.

<u>So:</u>

A is <u>not</u> diagonazible over the reals
$\Leftrightarrow$ There are <u>no</u> three (linearly) independent eigenvectors of A.

11. Consider the network of streets with intersections A, B, C, D, E and F below. The arrows indicate the direction of traffic flow along the **one-way streets**, and the numbers refer to the **exact** number of cars observed to enter or leave A, B, C, D, E and F during one minute. Each x_i denotes the unknown number of cars which passed along the indicated streets during the same period.



a) Write down a system of linear **equations** which describes the traffic flow, **together with all the constraints** on the variables x_i, $i = 1, \ldots, 6$. (Do not perform any operations on your equations: this is done for you in (b), and *do not simply copy out the equations implicit in (b). You will not get any marks if you do this.*)

	Flow in		Flow out
A	$x_6 + 70$	$=$	x_1
B	x_1	$=$	$x_2 + 10$
C	x_2	$=$	$x_3 + 35$
D	x_3	$=$	$x_4 + 75$
E	$x_4 + 20$	$=$	x_5
F	$x_5 + 30$	$=$	x_6

$x_1, \ldots, x_6 \in \mathbb{Z}$ with $x_1 \geq 0, \ldots, x_6 \geq 0$

11. **b)** The reduced row-echelon form of the augmented matrix of the system in part (a) is

$$\left[\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 0 & -1 & 70 \\ 0 & 1 & 0 & 0 & 0 & -1 & 60 \\ 0 & 0 & 1 & 0 & 0 & -1 & 25 \\ 0 & 0 & 0 & 1 & 0 & -1 & -50 \\ 0 & 0 & 0 & 0 & 1 & -1 & -30 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Give the general solution. (Ignore the constraints from (a) at this point.)

$$S = \left\{ \begin{pmatrix} 70+s \\ 60+s \\ 25+s \\ -50+s \\ -30+s \\ s \end{pmatrix} \middle| \; s \in \mathbb{R} \right\}$$

11. **c)** Find the minimum flow along BC, **using your results from (b).**
(You must justify all your answers.)

$\overline{BC}$ corresponds to x_2

By our restrictions, we have to ensure that all 6 entries are in $\mathbb{Z}$ and ≥ 0. This happens iff $s \in \mathbb{Z}$ and $s \geq 50$. Hence, the minimal flow along $\overline{BC}$ is 110.

12. Let $W = \{(x, y, z, u) \in \mathbf{R}^4 \mid x + y - u = 0\}$.

a) *Without referring to the Subspace Test* briefly explain why W is a subspace of $\mathbf{R}^4$. (Remember, I am not a high dimensional being, so geometric arguments in $\mathbf{R}^4$ won't convince me.)

b) Find a basis for W.

c) Use the Gram-Schmidt algorithm to find an orthogonal basis for W.
$2 = 1$ correct algorithm + .5 vectors their *are* orthogonal + .5 if all vectors actually belong to W

d) Find the best approximation by a vector in W to the vector $(1, 1, 1, 0)$.

(Remember: you must justify your answers.)

a)

$$W = \left\{ \begin{pmatrix} x \\ y \\ z \\ u \end{pmatrix} \in \mathbb{R}^4 \;\middle|\; x = -y + u \right\} = \left\{ \begin{pmatrix} -y+u \\ y \\ z \\ u \end{pmatrix} \;\middle|\; y, z, u \in \mathbb{R} \right\}$$

$$= \left\{ y \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + z \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + u \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \;\middle|\; y, z, u \in \mathbb{R} \right\}$$

$$= \operatorname{span} \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

... and spans are always subspaces ∇

b) The three vectors above are linearly independent because:

$$a \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + b \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + c \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow a = b = c = 0$$

So, $\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ forms a basis of W.

c)

$$w_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$w_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} - \frac{\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}}{\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}} \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} - \frac{0}{2} \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$w_3 = \begin{pmatrix}1\\0\\0\\1\end{pmatrix} - \frac{\begin{pmatrix}-1\\1\\0\\0\end{pmatrix}\cdot\begin{pmatrix}1\\0\\0\\1\end{pmatrix}}{\begin{pmatrix}-1\\1\\0\\0\end{pmatrix}\cdot\begin{pmatrix}-1\\1\\0\\0\end{pmatrix}}\begin{pmatrix}-1\\1\\0\\0\end{pmatrix} - \frac{\begin{pmatrix}0\\0\\1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\0\\0\\1\end{pmatrix}}{\begin{pmatrix}0\\0\\1\\0\end{pmatrix}\cdot\begin{pmatrix}0\\0\\1\\0\end{pmatrix}}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$$

$$= \begin{pmatrix}1\\0\\0\\1\end{pmatrix} - \frac{-1}{2}\begin{pmatrix}-1\\1\\0\\0\end{pmatrix} - \frac{0}{1}\begin{pmatrix}0\\0\\1\\0\end{pmatrix} = \begin{pmatrix}1/2\\1/2\\0\\1\end{pmatrix}$$

Note: If you start with another basis, the resulting orthogonal basis may be different.

d)

$$\text{proj}_W\begin{pmatrix}1\\1\\1\\0\end{pmatrix} = \frac{\begin{pmatrix}-1\\1\\0\\0\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\\0\end{pmatrix}}{\begin{pmatrix}-1\\1\\0\\0\end{pmatrix}\cdot\begin{pmatrix}-1\\1\\0\\0\end{pmatrix}}\begin{pmatrix}-1\\1\\0\\0\end{pmatrix} + \frac{\begin{pmatrix}0\\0\\1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\\0\end{pmatrix}}{\begin{pmatrix}0\\0\\1\\0\end{pmatrix}\cdot\begin{pmatrix}0\\0\\1\\0\end{pmatrix}}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$$

$$+ \frac{\begin{pmatrix}1/2\\1/2\\0\\1\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\\0\end{pmatrix}}{\begin{pmatrix}1/2\\1/2\\0\\1\end{pmatrix}\cdot\begin{pmatrix}1/2\\1/2\\0\\1\end{pmatrix}}\begin{pmatrix}1/2\\1/2\\0\\1\end{pmatrix}$$

$$= \frac{0}{2}\begin{pmatrix}-1\\1\\0\\0\end{pmatrix} + \frac{1}{1}\begin{pmatrix}0\\0\\1\\0\end{pmatrix} + \underbrace{\frac{1}{3/2}\begin{pmatrix}1/2\\1/2\\0\\1\end{pmatrix}}_{\begin{pmatrix}1/3\\1/3\\0\\2/3\end{pmatrix}} = \begin{pmatrix}1/3\\1/3\\1\\2/3\end{pmatrix}$$

That's the best approximation.

13. Let $A = \begin{bmatrix} 1 & 0 & -2 \\ 1 & 3 & 1 \\ 1 & 0 & 4 \end{bmatrix}$.

a) Compute the characteristic polynomial of A and factor it to show that the only eigenvalues of A are 2 and 3.

b) Find a basis of $E_2 = \{v \in \mathbf{R}^3 \mid Av = 2v\}$.

c) Find a basis of $E_3 = \{v \in \mathbf{R}^3 \mid Av = 3v\}$.

d) If possible, find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$. If A is diagonalizable, explain why your choice of P is invertible. If A is not diagonalizable, explain why.

a) $$\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 0 & -2 \\ 1 & 3-\lambda & 1 \\ 1 & 0 & 4-\lambda \end{bmatrix}$$

2nd column expansion

$$= (3-\lambda) \cdot \det \begin{bmatrix} 1-\lambda & -2 \\ 1 & 4-\lambda \end{bmatrix}$$

alternatively: $= -\lambda^3 + 8\lambda^2 - 21\lambda + 18$

$$= (3-\lambda) \cdot \left((1-\lambda)(4-\lambda) + 2 \right)$$

$$= (3-\lambda) \cdot (\lambda^2 - 5\lambda + 6)$$

$$= (3-\lambda) \cdot (\lambda - 3) \cdot (\lambda - 2)$$

$$= -(\lambda-3)^2 (\lambda - 2)$$

already factorized

"characteristic polynomial"

Hence, $\lambda = 3$ and $\lambda = 2$ are the eigenvalues of A.

b) $$E_2 = \ker(A - 2I) = \ker \begin{bmatrix} -1 & 0 & -2 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$$

We solve the respective homogeneous linear system:

$$\left[\begin{array}{ccc|c} -1 & 0 & -2 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \text{RREF}$$

$$E_2 = S = \left\{ \begin{pmatrix} -2s \\ s \\ s \end{pmatrix} \middle| \, s \in \mathbb{R} \right\} = \left\{ s \cdot \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \middle| \, s \in \mathbb{R} \right\}$$

$$= \text{span} \underbrace{\left\{ \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \right\}}_{\text{linearly independent}} \qquad \underline{\text{basis:}} \quad \left\{ \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \right\}$$

c) $E_3 = \ker(A - 3I) = \ker \begin{bmatrix} -2 & 0 & -2 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ As above:

$$\left[\begin{array}{ccc|c} -2 & 0 & -2 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \text{ RREF}$$

$$E_3 = S = \left\{ \begin{pmatrix} -t \\ s \\ t \end{pmatrix} \middle| \, s, t \in \mathbb{R} \right\} = \left\{ s \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + t \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \middle| \, s, t \in \mathbb{R} \right\}$$

$$= \text{span} \underbrace{\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}}_{\text{linearly independent}} \qquad \underline{\text{basis:}} \quad \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

d) $P := \begin{bmatrix} -2 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ $\qquad P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

basis of E_3

basis of E_2

P is invertible because $\det(P) = 1 \cdot \det \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix} = 1 \cdot (-1) = -1 \neq 0$.

14. Define a linear transformation $S : \mathbf{R}^3 \to \mathbf{R}^3$ by

$$S(\begin{bmatrix} x \\ y \\ z \end{bmatrix}) = \begin{bmatrix} x+y \\ y+z \\ z-x \end{bmatrix}.$$

a) Find the standard matrix of S.

b) Find a basis for im S.

c) Give a complete geometric description of im S.

d) Find the dimension of $\ker S$.

We won't cover §24 this year!

15 (a). State whether each of the following is (always) true, or is (possibly) false, in the box after the statement.

- If you say the statement may be false, you **must give an explicit example - with numbers - to show this.**
- If you say the statement is true, you must give a clear explanation - by quoting a theorem presented in class, or **by giving a proof valid for every case**.

(i) If A is a 2 by 2 matrix and A is diagonalizable, then A is invertible.

FALSE

$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is diagonalizable because $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are both eigenvectors (with corresponding eigenvalue 0), and they form a basis of $\mathbb{R}^2$.

But, $\text{rank}(A) = 0$, so A is not invertible.

(ii) Suppose B is an invertible 3×3 matrix, and that $\{v_1, v_2, v_3\}$ spans $\mathbf{R}^3$. Then $\{Bv_1, Bv_2, Bv_3\}$ also spans $\mathbf{R}^3$.

TRUE

Let v be an arbitrary vector in $\mathbb{R}^3$.
We have to show that there are scalars $a, b, c \in \mathbb{R}$ such that:

$$a \cdot Bv_1 + b \cdot Bv_2 + c \cdot Bv_3 = v$$

$$\Leftrightarrow \quad B^{-1}(aBv_1 + bBv_2 + cBv_3) = B^{-1}v$$

$$\Leftrightarrow \quad a\,B^{-1}B\,v_1 + b\,B^{-1}B\,v_2 + c\,B^{-1}B\,v_3 = B^{-1}v$$

$$\Leftrightarrow \quad a v_1 + b v_2 + c v_3 = B^{-1}v$$

But, since $\{v_1, v_2, v_3\}$ spans $\mathbb{R}^3$, we can find appropriate scalars $a, b, c \in \mathbb{R}$ to solve the last and, hence, the first equation.

15 (b). Let A be a real $n \times n$ matrix. Give 3 statements (in total) equivalent to

"A is **not** invertible",

one each in terms of:

(I) the columns of A

are linearly dependent.

(II) the determinant of A

is equal to 0.

(III) the homogeneous linear system $Ax = 0$, where $x \in \mathbf{R}^n$

has infinitely many solutions.

16. (4 bonus marks) **Make sure you finish and check the rest of the paper before trying this. Bonus marks are much harder to earn.**

Suppose $\{u, v, w\} \subset \mathbf{R}^3$ is a set of non-zero vectors.

Prove the following carefully. **N.B. Your proof must be valid for <u>every</u> set $\{u, v, w\}$ of non-zero vectors in $\mathbf{R}^3$, so don't choose any of them yourself!**

If $|u \cdot v \times w| = \|u\|\,\|v\|\,\|w\|$, then $\{u, v, w\}$ is an orthogonal set.

Recall the following two formulae:

For any two vectors $v_1, v_2 \in \mathbb{R}^3$

(i) $v_1 \cdot v_2 = \|v_1\| \cdot \|v_2\| \cdot \cos(\vartheta)$ VSF p.20

dot product

$0 \leq \vartheta \leq \pi$ angle between v_1 and v_2

(ii) $\|v_1 \times v_2\| = \|v_1\| \cdot \|v_2\| \cdot \sin(\vartheta)$ VSF p. 31

Cool, let's apply them!

$$|u \cdot v \times w| \overset{(i)}{=} \|u\| \cdot \|v \times w\| \cdot \cos(\vartheta_1)$$

where ϑ_1 is the angle between u and $v \times w$

$$\overset{(ii)}{=} \|u\| \cdot \|v\| \cdot \|w\| \cdot \sin(\vartheta_2) \cdot \cos(\vartheta_1)$$

where ϑ_2 is the angle between v and w

By our assumption,

$$\sin(\vartheta_2) \cdot \cos(\vartheta_1) = 1$$

Since ϑ_1 and ϑ_2 are both between 0 and π, this implies that $\sin(\vartheta_2) = 1$ and $\cos(\vartheta_1) = 1$. Since $\sin(\vartheta_2) = 1$, we know that $\vartheta_2 = \pi/2$. So, $v \perp w$.

By VSF p. 32, $|u \cdot v \times w| = |v \cdot w \times u| = |w \cdot u \times v|$, which implies that also $w \perp u$ and $u \perp v$.

(You may use this page for rough work or solutions that did not fit on previous pages.)