Faculty of Science »
Department of
mathematics and statistics

MAT 1341C Winter 2015 Final Exam

16 April, 2015.

Instructor - Barry Jessup

Family Name:____________________

First Name:____________________ Seat number: ________

Student number:____________________

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Some Advice

Take a few minutes to read the entire paper before you begin to write, and read each question carefully. The multiple choice questions are only worth 1 point and questions 11-15 are worth 6 points each. Make a note of the questions you feel confident you can do, and try those first: <u>you do not have to do the questions in the order they are presented</u>.

Instructions

1. You have 3 hours to complete this exam.
2. This is a closed book exam, and no notes of any kind are permitted. **The use of calculators, cell phones, or similar devices is not permitted.** All cyber devices not necessary for life-support must be disabled at the beginning of the exam.
3. Questions 1 to 10 are multiple choice. These questions have just one correct answer, are worth <u>1 point</u> each and no part marks will be given. Please record your answers in the spaces opposite.
4. Questions 11 – 15 require a complete solution, and are worth <u>6 points</u> each. Question 16 is a bonus question and should only be attempted after all other questions have been completed and checked.

   Spend your time accordingly.

   Answer questions 11 – 16 in the space provided, and use the backs of pages if necessary.
5. **The correct answer requires justification written legibly and logically: you must convince me that you know why your solution is correct.**
6. Where it is possible to check your work, do so.

Good luck! Bonne chance!

1. Let $X = \{(x, y, z) \in \mathbf{R}^3 \mid x - 2y + z = 0\}$. Which one of the following statements is true?

A. X is a subspace of $\mathbf{R}^3$ and $\dim X = 3$

B. X is a subspace of $\mathbf{R}^3$ and $\{(2, 1, 0), (-1, 0, 1)\}$ is a basis of X

C. X is a subspace of $\mathbf{R}^3$ and $\{(2, 1, 0), (4, 2, 0)\}$ is a basis of X

D. X is a not a subspace of $\mathbf{R}^3$.

E. X is a line in $\mathbf{R}^3$ with direction vector $(1, -2, 1)$

F. X is a plane in $\mathbf{R}^3$ with normal $(2, 1, 0)$

<u>Note that</u> $X = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \,\middle|\, x = 2y - z \right\} = \left\{ \begin{pmatrix} 2y-z \\ y \\ z \end{pmatrix} \,\middle|\, y, z \in \mathbb{R} \right\}$

$= \left\{ y \cdot \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + z \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \,\middle|\, y, z \in \mathbb{R} \right\} = \text{span} \underbrace{\left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}}_{\text{linearly independent}}$

2. Which two of the following statements are true?

I. $\{1, x, x^2\}$ is linearly independent in $\mathbf{F}(\mathbf{R}) = \{f \mid f : \mathbf{R} \to \mathbf{R}\}$.
II. A homogeneous linear system always has infinitely many solutions.
III. If A and B are 2×2 matrices, and A is invertible then $AB = 0$ implies $B = 0$.
IV. If u and v are independent vectors in $\mathbf{R}^3$, then $\{u, v\} = \text{span}\{u, v\}$.

A. I and II.
B. I and III.
C. I and IV.
D. II and III.
E. II and IV.
F. III and IV.

I $\quad a \cdot 1 + b \cdot x + c \cdot x^2 = 0$

$\left.\begin{array}{ll} \boxed{x=0} & a = 0 \\ \boxed{x=1} & a + b + c = 0 \\ \boxed{x=-1} & a - b + c = 0 \end{array}\right\} a = b = c = 0$

So, $\{1, x, x^2\}$ is linearly independent.

II No! The system with augmented matrix $\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$ hasn't.

III Yes! $AB = 0 \Rightarrow A^{-1}AB = A^{-1}0 \Rightarrow B = 0$.

IV No! Consider $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. Then $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$ consists of two vectors only whereas $\text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$ consists of infinitely many vectors.

3. Let A be a square $n \times n$ matrix with $n \geq 2$. Which of the following statements is true?

I. If rank $A = n-1$, there is just one parameter in the general solution of $Ax = 0$.
II. If rank $A = n-1$, there are $n-1$ parameters in the general solution of $Ax = 0$.
III. If A is invertible, $Ax = 0$ has more than one solution
IV. If $Ax = 0$ has more than one solution, then rank $A < n$.

A. I. only
B. II. only
C. III only
D. I. and III.
E. I and IV.
F. III and IV.

I Yes! In this case, the RREF of A has a leading one in all but one column. The remaining column without leading one corresponds to a free parameter.

II Well, this would only be true for $n=2$ (by the same argument as in I). But, not in general.

III No! $Ax = 0 \Rightarrow A^{-1}Ax = A^{-1}0 \Rightarrow x = 0$. So, there is only one solution, namely the trivial one $x=0$.

IV Yes! If $Ax=0$ has more than one solution, there must be a free parameter. This is only possible if not every column has a leading one, that is, if $\text{rank}(A) < n$.

4. If $A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$ and D is a $2 \times n$ matrix, then the second row of the matrix AD is

A. not defined unless $n = 2$.
B. twice the first row of D.
C. the same as the first row of D.
D. the same as the second row of D.
E. the sum of the first and the second rows of D.
F. the sum of twice the first row of D and the second row of D.

$$\underbrace{\begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} \vdots & \vdots & \cdots & \vdots \end{bmatrix}}_{D} = \dots$$

Since the second row of A is $[1 \ 1]$, the second row of AD is the sum of the first and the second row of D.

5. Let $S = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$ and consider the subset $W = \{A \in \mathbf{M}_{22} \mid SA = AS\}$. Which one of the following statements is true?

A. W is not a subspace of $\mathbf{M}_{22}$
B. W is a subspace of $\mathbf{M}_{22}$, and $\dim W = 4$
C. W is a subspace of $\mathbf{M}_{22}$, and $\dim W = 3$
D. W is a subspace of $\mathbf{M}_{22}$, and $\dim W = 2$
E. W is a subspace of $\mathbf{M}_{22}$, and $\dim W = 1$
F. W is a subspace of $\mathbf{M}_{22}$, and $\dim W = 0$

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then,

$$SA = AS$$

$$\Leftrightarrow \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} a+2c & b+2d \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & 2a \\ c & 2c \end{bmatrix}$$

$$\Leftrightarrow \begin{aligned} a+2c &= a \\ b+2d &= 2a \\ 0 &= c \\ 0 &= 2c \end{aligned}$$

$$\Leftrightarrow b = 2a - 2d \ \underline{\text{and}} \ c = 0$$

So,

$$W = \left\{ \begin{bmatrix} a & 2a-2d \\ 0 & d \end{bmatrix} \middle| \ a, d \in \mathbb{R} \right\}.$$

We may rewrite this:

$$W = \left\{ a \cdot \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} + d \cdot \begin{bmatrix} 0 & -2 \\ 0 & 1 \end{bmatrix} \middle| \ a, d \in \mathbb{R} \right\}$$

$$= \operatorname{span} \underbrace{\left\{ \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -2 \\ 0 & 1 \end{bmatrix} \right\}}_{\text{linearly independent}}$$

Recall that spans are always subspaces!!!

6. Find the value of t for which $(1, 3, t)$ belongs to $\operatorname{span}\{(1,2,1), (1,1,2)\}$.

A. -2
B. -1
C. 0
D. 1
E. 2
F. 7

There are many ways to solve this. Here is the most elementary one:

$$a \cdot \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + b \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ t \end{pmatrix}$$

This has a solution if and only if the following augmented matrix corresponds to a consistent linear system:

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & 1 & 3 \\ 1 & 2 & t \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 1 & t-1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & t-1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & t \end{array}\right]$$

Consistent $\Leftrightarrow t = 0$

7. For a non-homogeneous system of 15 equations in 12 unknowns, answer the following three questions:

- Can the system be inconsistent?
- Can the system have a unique solution?
- Can the system have infinitely many solutions?

A. No, Yes, Yes.
B. Yes, Yes, No.
C. Yes, No, Yes.
D. No, No, Yes.
E. Yes, Yes, Yes.
F. No, Yes, No.

not all zero

$$15 \left\{ \underbrace{\left[\begin{array}{ccc|c} & & & \\ & & & \\ & & & \end{array}\right]}_{12} \right.$$

It can be inconsistent, e.g.:

$$\left[\begin{array}{ccc|c} 0 & \cdots & 0 & 1 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 1 \end{array}\right]$$

It can have a unique solution, e.g.:

$$\left[\begin{array}{ccc|c} 1 & & & 1 \\ & \ddots & & \vdots \\ & & 1 & 1 \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 \end{array}\right]$$

It can have infinitely many solutions, e.g.:

$$\left[\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{array}\right]$$

8. Let A be a square $n \times n$ matrix. Which one of the statements below **is not equivalent** to

"The rows of A are linearly independent"

A. 0 is an eigenvalue of A

B. The homogeneous system $Ax = 0$ has a unique solution

C. The columns of A form a basis of $\mathbf{R}^n$

D. rank $A = n$

E. $\det A \neq 0$

F. A is invertible

By our big translation theorem!

It's A! If 0 is an eigenvalue of A, then there is a non-zero vector x such that $Ax = 0 \cdot x = 0$. So, there is a non-zero vector in the kernel of A. So, the rows of A can't be linearly independent.

Example $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ has eigenvalue 0 because $A \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

But the rows of A are not linearly independent.

9. If $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -3$, find

$$\begin{vmatrix} 3a & 3d & 3g \\ b+5c & e+5f & h+5i \\ -2c & -2f & -2i \end{vmatrix}$$

A. -64
B. -18
C. -16
D. 16
E. 18
F. 64

$$\det\begin{bmatrix} 3a & 3d & 3g \\ b+5c & e+5f & h+5i \\ -2c & -2f & -2i \end{bmatrix} = 3\cdot(-2)\cdot\det\begin{bmatrix} a & d & g \\ b+5c & e+5f & h+5i \\ c & f & i \end{bmatrix}$$

$$= 3\cdot(-2)\cdot\det\begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}$$

$$= 3\cdot(-2)\cdot\det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

transpose: $\det(A^T) = \det(A)$

$$= 3\cdot(-2)\cdot(-3) = 18$$

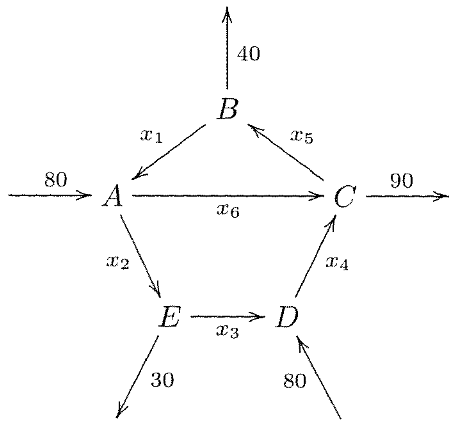
10. The vectors $u_1 = (1,-1,2), u_2 = (-5,-1,2)$, and $u_3 = (0,2,1)$ form an orthogonal basis of $\mathbf{R}^3$. If we write $v = (1,0,-1) = a_1u_1 + a_2u_2 + a_3u_3$, what is a_2?

A. $-\frac{1}{3}$
B. $\frac{1}{3}$
C. $-\frac{7}{30}$
D. $\frac{7}{30}$
E. $-\frac{1}{5}$
F. $\frac{1}{5}$

$$a_2 = \frac{u_2 \cdot v}{u_2 \cdot u_2}$$

$$= \frac{\begin{pmatrix} -5 \\ -1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}{\begin{pmatrix} -5 \\ -1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -5 \\ -1 \\ 2 \end{pmatrix}} = \frac{-7}{30} = -\frac{7}{30}$$

11. Consider the network of streets with intersections A, B, C, D and E below. The arrows indicate the direction of traffic flow along the **one-way streets**, and the numbers refer to the **exact** number of cars observed to enter or leave A, B, C, D and E during one minute. Each x_i denotes the unknown number of cars which passed along the indicated streets during the same period.



a) Write down a system of linear equations which describes the the traffic flow, together with all the constraints on the variables x_i, $i = 1, \ldots, 6$.
(Do not perform any operations on your equations: this is done for you in (b). Do not simply copy out the equations implicit in (b). You will not get any marks if you do this.)

	Flow in		Flow out
A	$x_1 + 80$	$=$	$x_2 + x_6$
B	x_5	$=$	$x_1 + 40$
C	$x_4 + x_6$	$=$	$x_5 + 90$
D	$x_3 + 80$	$=$	x_4
E	x_2	$=$	$x_3 + 30$

$x_1, \ldots, x_6 \in \mathbb{Z}$ with $x_1 \geqslant 0, \ldots, x_6 \geqslant 0$

11(b). The reduced row-echelon form of the augmented matrix of the system in part (a) is

$$\left[\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & -1 & 0 & -40 \\ 0 & 1 & 0 & 0 & -1 & 1 & 40 \\ 0 & 0 & 1 & 0 & -1 & 1 & 10 \\ 0 & 0 & 0 & 1 & -1 & 1 & 90 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Give the general solution. (Ignore the constraints from (a) at this point.)

$$S = \left\{ \begin{pmatrix} -40+s \\ 40+s-t \\ 10+s-t \\ 90+s-t \\ s \\ t \end{pmatrix} \,\middle|\, s,t \in \mathbb{R} \right\}$$

c) If $\overline{AC}$ were closed due to roadwork, find the minimum flow along $\overline{ED}$, **using your results from (b).**

(You must justify all your answers.)

$\overline{AC}$ corresponds to $x_6 = t$

$\overline{ED}$ corresponds to x_3

If $x_6 = 0$, then

$$S' = \left\{ \begin{pmatrix} -40+s \\ 40+s \\ 10+s \\ 90+s \\ s \\ 0 \end{pmatrix} \,\middle|\, s \in \mathbb{R} \right\}$$

By our restrictions, we have to ensure that all 6 entries are in $\mathbb{Z}$ and ≥ 0. This happens iff $s \in \mathbb{Z}$ and $s \geq 40$. Hence, the minimal flow along $\overline{ED}$ is 50.

12. Let $X = \text{span}\{(1,-1,1,0),(0,1,1,1),(1,2,4,3),(1,0,2,2)\}$.

a) Find any basis for X, and hence find $\dim X$.

b) Find a basis for X which is a **subset** of the given spanning set above.

c) Extend your basis for X in part (b) to a basis of $\mathbf{R}^4$.

d) If X were the rowspace of a 4×4 matrix A, how many parameters would there be in the general solution to $Ax = 0$?

a) We run the row space algorithm:

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 2 & 4 & 3 \\ 1 & 0 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 3 & 3 & 3 \\ 0 & 1 & 1 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{REF}$$

A basis is: $\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

b) Now, we need to run the column space algorithm (which we could also have done in a); so better read all questions before answering them):

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ -1 & 1 & 2 & 0 \\ 1 & 1 & 4 & 2 \\ 0 & 1 & 3 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 3 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{REF}$$

So, a basis with the desired property consists of the 1^{st}, the 2^{nd}, and the 4^{th} vector of the original spanning set:

$$\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \\ 2 \end{pmatrix} \right\}$$

c) Now, we take the three vectors above and check which columns don't have a leading one after running the row space algorithm:

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 2 & 2 \end{bmatrix} \sim \underset{\text{as in a)}}{\cdots} \sim \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{REF}$$

There is no leading one in the third column, so we have to add $(0,0,1,0)$. Hence, we obtain:

$$\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

d) There would be one free parameter, as it becomes clear when considering our computations in a) and augmenting the matrix with a column full of zeros.

Alternative argument:

$$\begin{aligned} \#\text{ free parameters} &= \dim \ker(A) \\ &= \#\text{ columns of } A - \operatorname{rank}(A) \\ &= 4 - 3 = 1 \end{aligned}$$

13. Let $W = \{(x, y, z, u) \in \mathbf{R}^4 \mid x - z - u = 0\}$.

a) *Without referring to the Subspace Test* briefly explain why W is a subspace of $\mathbf{R}^4$.

b) Find a basis for W.

c) Use the **Gram-Schmidt algorithm** to find an orthogonal basis for W.

d) Find the best approximation by a vector in W to the vector $(1, 0, 1, 1)$.

a) $$W = \left\{ \begin{pmatrix} x \\ y \\ z \\ u \end{pmatrix} \in \mathbb{R}^4 \;\middle|\; x = z + u \right\} = \left\{ \begin{pmatrix} z+u \\ y \\ z \\ u \end{pmatrix} \;\middle|\; y, z, u \in \mathbb{R} \right\}$$

$$= \left\{ y \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + z \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + u \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \;\middle|\; y, z, u \in \mathbb{R} \right\}$$

$$= \operatorname{Span} \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

... and spans are always subspaces ∇

b) The three vectors above are linearly independent because:

$$a \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + b \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + c \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow a = b = c = 0$$

So, $\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ forms a basis of W.

c) $$w_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$w_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} - \frac{\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}}{\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} - \frac{0}{1} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$w_3 = \begin{pmatrix}1\\0\\0\\1\end{pmatrix} - \frac{\begin{pmatrix}0\\1\\0\\0\end{pmatrix}\cdot\begin{pmatrix}1\\0\\0\\1\end{pmatrix}}{\begin{pmatrix}0\\1\\0\\0\end{pmatrix}\cdot\begin{pmatrix}0\\1\\0\\0\end{pmatrix}}\begin{pmatrix}0\\1\\0\\0\end{pmatrix} - \frac{\begin{pmatrix}1\\0\\1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\0\\0\\1\end{pmatrix}}{\begin{pmatrix}1\\0\\1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\0\\1\\0\end{pmatrix}}\begin{pmatrix}1\\0\\1\\0\end{pmatrix}$$

$$= \begin{pmatrix}1\\0\\0\\1\end{pmatrix} - \frac{0}{1}\begin{pmatrix}0\\1\\0\\0\end{pmatrix} - \frac{1}{2}\begin{pmatrix}1\\0\\1\\0\end{pmatrix} = \begin{pmatrix}1/2\\0\\-1/2\\1\end{pmatrix}$$

An orthogonal basis for W is:

$$\left\{ \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}1/2\\0\\-1/2\\1\end{pmatrix} \right\}$$

Note: If you start with another basis, the resulting orthogonal basis may be different.

d)

$$\text{proj}_W \begin{pmatrix}1\\0\\1\\1\end{pmatrix} = \frac{\begin{pmatrix}0\\1\\0\\0\end{pmatrix}\cdot\begin{pmatrix}1\\0\\1\\1\end{pmatrix}}{\begin{pmatrix}0\\1\\0\\0\end{pmatrix}\cdot\begin{pmatrix}0\\1\\0\\0\end{pmatrix}}\begin{pmatrix}0\\1\\0\\0\end{pmatrix} + \frac{\begin{pmatrix}1\\0\\1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\0\\1\\1\end{pmatrix}}{\begin{pmatrix}1\\0\\1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\0\\1\\0\end{pmatrix}}\begin{pmatrix}1\\0\\1\\0\end{pmatrix} + \frac{\begin{pmatrix}1/2\\0\\-1/2\\1\end{pmatrix}\cdot\begin{pmatrix}1\\0\\1\\1\end{pmatrix}}{\begin{pmatrix}1/2\\0\\-1/2\\1\end{pmatrix}\cdot\begin{pmatrix}1/2\\0\\-1/2\\1\end{pmatrix}}\begin{pmatrix}1/2\\0\\-1/2\\1\end{pmatrix}$$

$$= \frac{0}{1}\begin{pmatrix}0\\1\\0\\0\end{pmatrix} + \frac{2}{2}\begin{pmatrix}1\\0\\1\\0\end{pmatrix} + \underbrace{\frac{1}{3/2}\cdot\begin{pmatrix}1/2\\0\\-1/2\\1\end{pmatrix}}_{\begin{pmatrix}1/3\\0\\-1/3\\2/3\end{pmatrix}} = \begin{pmatrix}4/3\\0\\2/3\\2/3\end{pmatrix}$$

That's the best approximation.

not easy for this A !

14. Let $A = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$.

a) Find the characteristic polynomial of A, and use this to show that the eigenvalues of A are 2 and -1.

b) Find a basis of $E_2 = \{v \in \mathbf{R}^3 \mid Av = 2v\}$.

c) Find a basis of $E_{-1} = \{v \in \mathbf{R}^3 \mid Av = -v\}$.

d) If possible, find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$. If this is not possible, explain why.

a) $\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & -1 & -1 \\ -1 & 1-\lambda & -1 \\ -1 & -1 & 1-\lambda \end{bmatrix}$

1st row expansion

$$= (1-\lambda) \cdot \det \begin{bmatrix} 1-\lambda & -1 \\ -1 & 1-\lambda \end{bmatrix} - (-1) \cdot \det \begin{bmatrix} -1 & -1 \\ -1 & 1-\lambda \end{bmatrix} + (-1) \cdot \det \begin{bmatrix} -1 & 1-\lambda \\ -1 & -1 \end{bmatrix}$$

$$= (1-\lambda) \cdot \left((1-\lambda)(1-\lambda) - 1\right) + \left(-(1-\lambda) - 1\right) - \left(1 + (1-\lambda)\right)$$

$$= (1-\lambda) \cdot (\lambda^2 - 2\lambda) + (\lambda - 2) - (-\lambda + 2)$$

$$= \lambda^2 - 2\lambda - \lambda^3 + 2\lambda^2 + \lambda - 2 + \lambda - 2$$

$$= -\lambda^3 + 3\lambda^2 - 4 \quad \text{"characteristic polynomial"}$$

Factorize:

$$\stackrel{!}{=} -(\lambda - 2)^a \cdot (\lambda + 1)^b \quad \text{with } a, b \geq 1 \text{ and } a + b = 3$$

...because someone tells us that 2 and -1 are potential eigenvalues !

This is necessary to obtain $-\lambda^3$ as highest term !

...because the polynomial has degree 3 !

It is now easy to verify that $a = 2$ and $b = 1$ do the job. Indeed:

$$-(\lambda-2)^2\cdot(\lambda+1) = -(\lambda-2)(\lambda-2)(\lambda+1)$$
$$= -(\lambda^2-4\lambda+4)(\lambda+1)$$
$$= -(\lambda^3-4\lambda^2+4\lambda+\lambda^2-4\lambda+4)$$
$$= -(\lambda^3-3\lambda^2+4) = \underline{\underline{-\lambda^3+3\lambda^2-4}}$$

So,

$$\det(A-\lambda I) = -(\lambda-2)^2\cdot(\lambda+1)$$

Hence, $\lambda=2$ and $\lambda=-1$ are the eigenvalues of A.

<u>Remark</u> If you think that this way is not very elegant, you are right. We could alternatively have applied elementary row operations to produce a row with 2 zeros along which we could have expanded the determinant more easily:

$$\det(A-\lambda I) = \det\begin{bmatrix}1-\lambda & -1 & -1\\ -1 & 1-\lambda & -1\\ -1 & -1 & 1-\lambda\end{bmatrix}$$

$$\overset{R_3-R_1\to R_3}{=} \det\begin{bmatrix}1-\lambda & -1 & -1\\ -1 & 1-\lambda & -1\\ -2+\lambda & 0 & 2-\lambda\end{bmatrix}$$

$$\overset{C_3+C_1\to C_3}{=} \det\begin{bmatrix}1-\lambda & -1 & -\lambda\\ -1 & 1-\lambda & -2\\ -2+\lambda & 0 & 0\end{bmatrix}$$

$$\overset{\text{3rd row expansion}}{=} (-2+\lambda)\cdot\det\begin{bmatrix}-1 & -\lambda\\ 1-\lambda & -2\end{bmatrix}$$

$$= (\lambda-2)\cdot\big(2+\lambda(1-\lambda)\big)$$

$$= (\lambda-2)\cdot(-\lambda^2+\lambda+2)$$

$$= (\lambda-2)\cdot(-1)\cdot(\lambda-2)\cdot(\lambda+1)$$

$$= -(\lambda-2)^2\cdot(\lambda+1).$$ ☺

A third alternative would have been to factorize $-\lambda^3+3\lambda^2-4$ by observing that $\lambda=2$ is a root and then running long division of polynomials to obtain $(\lambda-2)\cdot(-\lambda^2+\lambda+2)$, which can be factorized more easily.

b) $E_2 = \ker(A-2I) = \ker\begin{bmatrix}-1&-1&-1\\-1&-1&-1\\-1&-1&-1\end{bmatrix}$

We solve the respective homogeneous linear system:

$$\left[\begin{array}{ccc|c}-1&-1&-1&0\\-1&-1&-1&0\\-1&-1&-1&0\end{array}\right] \sim \left[\begin{array}{ccc|c}1&1&1&0\\-1&-1&-1&0\\-1&-1&-1&0\end{array}\right] \sim \left[\begin{array}{ccc|c}1&1&1&0\\0&0&0&0\\0&0&0&0\end{array}\right] \text{ RREF}$$

$$E_2 = S = \left\{\begin{pmatrix}-s-t\\s\\t\end{pmatrix} \,\middle|\, s,t\in\mathbb{R}\right\} = \left\{s\cdot\begin{pmatrix}-1\\1\\0\end{pmatrix}+t\cdot\begin{pmatrix}-1\\0\\1\end{pmatrix} \,\middle|\, s,t\in\mathbb{R}\right\}$$

$$= \operatorname{span}\underbrace{\left\{\begin{pmatrix}-1\\1\\0\end{pmatrix},\begin{pmatrix}-1\\0\\1\end{pmatrix}\right\}}_{\text{linearly independent}} \qquad \underline{\text{basis:}}\ \left\{\begin{pmatrix}-1\\1\\0\end{pmatrix},\begin{pmatrix}-1\\0\\1\end{pmatrix}\right\}$$

c) $E_{-1} = \ker(A+I) = \ker\begin{bmatrix}2&-1&-1\\-1&2&-1\\-1&-1&2\end{bmatrix}$. As above:

$$\left[\begin{array}{ccc|c}2&-1&-1&0\\-1&2&-1&0\\-1&-1&2&0\end{array}\right] \sim \left[\begin{array}{ccc|c}-1&2&-1&0\\2&-1&-1&0\\-1&-1&2&0\end{array}\right] \sim \left[\begin{array}{ccc|c}1&-2&1&0\\2&-1&-1&0\\-1&-1&2&0\end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c}1&-2&1&0\\0&3&-3&0\\0&-3&3&0\end{array}\right] \sim \left[\begin{array}{ccc|c}1&-2&1&0\\0&1&-1&0\\0&0&0&0\end{array}\right] \sim \left[\begin{array}{ccc|c}1&0&-1&0\\0&1&-1&0\\0&0&0&0\end{array}\right] \text{ RREF}$$

$$E_{-1} = S' = \left\{ \begin{pmatrix} s \\ s \\ s \end{pmatrix} \middle| \, s \in \mathbb{R} \right\} = \left\{ s \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \middle| \, s \in \mathbb{R} \right\}$$

$$= \operatorname{Span} \underbrace{\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}}_{\text{linearly independent}} \qquad \underline{\text{basis:}} \quad \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

d) $$P := \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \qquad P^{-1} A P = \underbrace{\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}}_{D}$$

basis of E_{-1}

basis of E_2

<u>Note that P is actually invertible:</u>

$$\det(P) = (-1) \cdot \det \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} - 1 \cdot \det \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

1^{st} column expansion

$$= (-1) \cdot (-1) - 1 \cdot (-2)$$

$$= 3 \neq 0 .$$

15. State whether each of the following is (always) true, or is (possibly) false, in the box after the statement.

- If you say the statement may be false, you **must give an explicit example - with numbers!**
- If you say the statement is true, you must give a clear explanation - by quoting a theorem presented in class, or by giving a *proof valid for every case.*

a) If A is an 4×3 matrix and if a row echelon form of A has a row of zeros, then rank $A < 3$.

Consider, for example:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$ RREF with a row of zeros

But rank$(A) = 3$

ANSWER $\boxed{\text{FALSE}}$

b) The dimension of the kernel of the matrix $[1 \quad 2 \quad 3 \quad 4 \quad 5]$ is 4.

$\dim \ker [1 \ 2 \ 3 \ 4 \ 5]$

$= \#$ columns of $[1 \ 2 \ 3 \ 4 \ 5]$ $-$ rank $[1 \ 2 \ 3 \ 4 \ 5]$ ← Rank-Nullity-Theorem

$= 5 - 1 = 4$

ANSWER $\boxed{\text{TRUE}}$

15 (cont.)

c) If $\{v_1, v_2\}$ is linearly independent in a vector space V, then $\{v_1 - v_2, v_1 + 2v_2\}$ is also linearly independent.

$$a \cdot (v_1 - v_2) + b \cdot (v_1 + 2v_2) = 0$$

$$\Rightarrow a v_1 - a v_2 + b v_1 + 2b v_2 = 0$$

$$\Rightarrow (a+b) v_1 + (-a+2b) v_2 = 0$$

Since $\{v_1, v_2\}$ is linearly independent, we know that

$$\begin{array}{l} a+b = 0 \\ -a+2b = 0 \end{array} \Rightarrow 3b = 0 \Rightarrow b = 0$$

So, also $a=0$. And we are done!

ANSWER TRUE

d) The funtion $T : \mathbf{R}^2 \to \mathbf{R}^2$ defined by $T(x, y) = (x^2, x+y)$ is a linear transformation.

We won't cover §24 this year!

ANSWER

16. (Four bonus marks) Make sure you finish and check the rest of the paper before trying this. As you know, bonus marks are much harder to earn.

In what follows, A denotes an $n \times n$ matrix.

a) Prove that if u, v and w are eigenvectors of A corresponding to three distinct eigenvalues, then $\{u, v, w\}$ is linearly independent.

Let $Au = \lambda_1 u$, $Av = \lambda_2 v$ $Aw = \lambda_3 w$. Then,

$$au + bv + cw \overset{(*)}{=} 0 \Rightarrow A \cdot (au + bv + cw) = A0$$

$$\Rightarrow a\lambda_1 u + b\lambda_2 v + c\lambda_3 w \overset{(**)}{=} 0$$

So, $(**) - \lambda_3 \cdot (*)$ yields: $a \cdot (\lambda_1 - \lambda_3) u + b \cdot (\lambda_2 - \lambda_3) v \overset{(\overset{*}{**})}{=} 0$

$$\Rightarrow A \cdot \left(a(\lambda_1 - \lambda_3) u + b(\lambda_2 - \lambda_3) v\right) = A0$$

$$\Rightarrow a(\lambda_1 - \lambda_3)\lambda_1 u + b(\lambda_2 - \lambda_3)\lambda_2 v \overset{(\overset{**}{**})}{=} 0$$

So, $(\overset{**}{**}) - \lambda_2 \cdot (\overset{*}{**})$ yields: $a \cdot \underbrace{(\lambda_1 - \lambda_3)}_{\neq 0} \underbrace{(\lambda_1 - \lambda_2)}_{\neq 0} \underset{\neq 0}{u} = 0$

Hence, $a = 0$.

b) Prove that if all the eigenvalues of A are non-zero, then A is invertible.

All eigenvalues of A are non-zero

$\Rightarrow E_0 = \{x \in \mathbb{R}^n \mid Ax = 0x\} = \{0\}$

$\Rightarrow \ker(A) = E_0 = \{0\}$

$\Rightarrow$ A is invertible.

So, $bv + cw \overset{(\Delta)}{=} 0 \Rightarrow A \cdot (bv + cw) = A0$

$$\Rightarrow b\lambda_2 v + c\lambda_3 w \overset{(\Delta\Delta)}{=} 0.$$

So, $(\Delta\Delta) - \lambda_3 \cdot (\Delta)$ yields: $b \underbrace{(\lambda_2 - \lambda_3)}_{\neq 0} \underset{\neq 0}{v} = 0$

Hence, $b = 0$.

So, $c\underset{\neq 0}{w} = 0$. Hence $c = 0$, and we are done!