MATH3705A - Test 3: 17:35–18:25, July 12, Tuesday

Surname _______________ **First Name** _______________**Student #** __________

Total points: 15. No partial marks for Questions 1-2.
Closed book! Non-programmer calculators are allowed!

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1. (4.5 points = 1.5+1.5+1.5) Let $f(x) = \left\{ \begin{array}{ll} 2, & 0 \le x \le 1; \\ x+3, & 1 < x < 3. \end{array} \right\}$.

   (i) What is the value of the Fourier series of $f(x)$ at $x = 1$?

   (a) 4 (b) 5 (c) 2 (d) 3 (e) 6

   Solution: (d).

   The extension $\tilde{f}(x)$ has period 3. The series converges to $\frac{\tilde{f}(1+)+\tilde{f}(1-)}{2} = \frac{4+2}{2} = 3$.

   (ii) What is the value of the Fourier **sine** series of $f(x)$ at $x = 10$?

   (a) -5 (b) -2 (c) -1 (d) 3 (e) 5

   Solution: (a)

   The extension $\tilde{f}$ has the period 6 and is odd. Since $10 = 6(2) - 2$, the Fourier sine series converges to

   $\tilde{f}_{av}(-2) = -\tilde{f}_{av}(2) = -f(2) = -5$.

   (iii) What is the value of the Fourier **cosine** series of $f(x)$ at $x = 2019$?

   (a) 4 (b) 5 (c) 2 (d) 3 (e) 6

   Solution: (d).

   The extension $\tilde{f}$ has the period 6 and is even. Since $2019 = 6(336) + 3$, the Fourier cosine series converges to

   $\tilde{f}_{av}(3) = \frac{\tilde{f}(3+)+\tilde{f}(3-)}{2} = \frac{6+6}{2} = 6$.

2. (1.5 points) Let $f(x) = 2 - x, 0 < x < 2$, and let a_n $(n \geq 0)$ be the coefficients of the Fourier **cosine** series. Find a_2.

(a) $\frac{2}{\pi^2}$ (b) $\frac{1}{\pi^2}$ (c) $\frac{1}{\pi}$ (d) $-\frac{1}{\pi}$ (e) 0

Solution: (e).

$$\begin{aligned} a_2 &= \frac{2}{L}\int_0^L f(x)\cos(\frac{2\pi x}{L})\,dx = \int_0^2 (2-x)\cos(\frac{2\pi x}{2})\,dx \\ &= \left[\frac{1}{\pi}(2-x)\sin(\pi x) - \frac{1}{\pi^2}\cos(\pi x)\right]_0^2 = 0. \end{aligned}$$

3. (4 points) Let $f(x) = \pi x$, $-1 \leq x < 1$, and let $f(x)$ be 2-periodic. Find the Fourier series.

Solution: Since the function $f(x)$ is odd, $a_n = 0$ for $n \geq 0$.

(1 point)

Note that $L = 1$,

$$b_n = \int_{-1}^{1} \pi x \sin(n\pi x)dx = \left[-\frac{1}{n}x\cos(n\pi x) + \frac{1}{n^2\pi}\sin(n\pi x)\right]\Bigg|_{-1}^{1}$$

$$= -\frac{2}{n}\cos(n\pi) = (-1)^{n+1}\frac{2}{n}.$$

(2 points)

Thus, the Fourier series is:

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2}{n}\sin(n\pi x).$$

(1 point)

4. (5 points) The solution of the heat equation $w_t = \alpha^2 w_{xx}$, $0 \leq x \leq L, t \geq 0$, subject to the boundary conditions $w(0,t) = 0$, $w(L,t) = 0$ has the form

$$w(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{\alpha n\pi}{L}\right)^2 t}.$$

Find the solution $u(x,t)$ of $u_t = 4u_{xx}$, subject to the boundary conditions $u(0,t) = 1$, $u(2,t) = 3$, and the initial condition $u(x,0) = x + 3$.

Solution: Here $\alpha = 2$, $L = 2$, $A = 1$, $B = 3$. Thus

$$v(x) = x + 1. \qquad (1\ \textit{point})$$

Let $w(x,t) = u(x,t) - v(x)$. Then

$$w_t = 4w_{xx}, \quad 0 \le x \le 2, t \ge 0$$

subject to

$$w(0,t) = w(2,t) = 0; \quad w(x,0) = u(x,0) - v(x) = 2.$$

Thus, for $n \ge 1$,

$$\begin{aligned} b_n &= \frac{2}{L}\int_0^L [f(x) - v(x)] \sin(\frac{n\pi x}{L})\, dx = \frac{2}{2}\int_0^2 (2) \sin(\frac{n\pi x}{2})\, dx \\ &= \left[-\frac{4}{n\pi} \cos(\frac{n\pi x}{2}) \right]_0^2 \\ &= -\frac{4}{n\pi}[\cos(n\pi) - 1] = \frac{4[(-1)^{n+1} + 1]}{n\pi}. \end{aligned}$$

(3 points)

$$u(x,t) = x + 1 + \sum_{n=1}^{\infty} \frac{4[(-1)^{n+1} + 1]}{n\pi} \sin\left(\frac{n\pi x}{2}\right) e^{-(n\pi)^2 t}. \quad (1\ \textit{point})$$

Table of Fourier Series

1. The Fourier series of a $2L$ periodic function f is given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right],$$

with

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx = \frac{1}{L} \int_{\alpha}^{\alpha+2L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx, \quad n \geq 0,$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx = \frac{1}{L} \int_{\alpha}^{\alpha+2L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx, \quad n \geq 1,$$

where α is any real number. If f is an odd function, then

$$a_n = 0 \quad \textit{and} \quad b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx \quad n \geq 1.$$

If f is an even function, then

$$b_n = 0 \quad \textit{and} \quad a_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx \quad n \geq 0.$$

2. The Fourier series of a function $f(x)$ defined on $[a, b]$ with $b - a = 2L$ is given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right],$$

with

$$a_n = \frac{1}{L} \int_a^b f(x) \cos\left(\frac{n\pi x}{L}\right) dx, \quad n \geq 0,$$

$$b_n = \frac{1}{L} \int_a^b f(x) \sin\left(\frac{n\pi x}{L}\right) dx, \quad n \geq 1.$$

If the $2L$-periodic extension $\tilde{f}$ of f to $\mathbb{R}$ is an odd function, then $a_n = 0$, and if $\tilde{f}$ is an even function, then $b_n = 0$.

3. The Fourier sine series of a function f defined on $[0, L]$ is given by

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right), \quad b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx, \quad n \geq 1.$$

4. The Fourier cosine series of a function f defined on $[0, L]$ is given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right), \quad a_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx, \quad n \geq 0.$$