STAT 3502

Solution for assignment # 2 Total mark=25

1. Two fair six-sided dice are tossed independently. Let M = maximum of the two tosses.
   a. [2] What is the pmf of M?
   b. [2] Determine the cdf of M and graph it.
   Sol:

   a. $p(1) = P(M = 1) = P[(1,1)] = \frac{1}{36}$

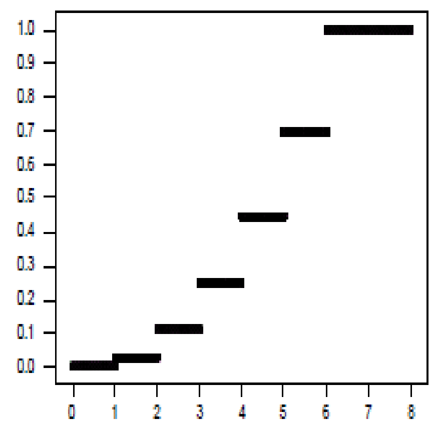
   $p(2) = P(M = 2) = P[(1,2) \text{ or } (2,1) \text{ or } (2,2)] = \frac{3}{36}$

   $p(3) = P(M = 3) = P[(1,3) \text{ or } (2,3) \text{ or } (3,1) \text{ or } (3,2) \text{ or } (3,3)] = \frac{5}{36}$

   Similarly, $p(4) = \frac{7}{36}$, $p(5) = \frac{9}{36}$, and $p(6) = \frac{11}{36}$

   b. $F(m) =$ 0 for $m < 1$, $\frac{1}{36}$ for $1 \le m < 2$,

$$F(m) = \begin{cases} 0 & m < 1 \\ \frac{1}{36} & 1 \le m < 2 \\ \frac{4}{36} & 2 \le m < 3 \\ \frac{9}{36} & 3 \le m < 4 \\ \frac{16}{36} & 4 \le m < 5 \\ \frac{25}{36} & 5 \le m < 6 \\ 1 & m \ge 6 \end{cases}$$



2. [2] Let x = the outcome when a fair die is rolled once. If before the die is rolled you are offered either $\frac{1}{3.5}$ dollars or $h(x) = \frac{1}{x}$ dollars , would you accept the guaranteed amount or would you gamble? Note: It is not generally true that $\frac{1}{E(X)} = E(\frac{1}{X})$
   Sol:

   $E[h(X)] = E\left(\frac{1}{X}\right) = \sum_{x=1}^{6}\left(\frac{1}{x}\right) \cdot p(x) = \frac{1}{6}\sum_{x=1}^{6}\frac{1}{x} = .408$, whereas $\frac{1}{3.5} = .286$, so you expect to win more if you gamble.

3. The weekly demand for propane gas (in 1000s of gallons) from a particular facility is a r.v. x with pdf

$$f(x) = \begin{cases} 2(1 - \frac{1}{x^2}) & 1 \le x \le 2 \\ 0 & \text{o.w.} \end{cases}$$

   a. [1.5] Compute the cdf of x.
   b. [1.5] What is the value of $\tilde{\mu}$ (median)?

c. [1.5] Compute $E(X)$ and $V(X)$.

d. [1.5] If 1.5 thousands gallons are in stock at the beginning of the week and no new supply is due in during the week, how much of the 1.5 thousands gallons is expected to be left at the end of the week. (Hint: Let $h(x)$ = amount left when demand = x.

Sol:

a. For $1 \le x \le 2$, $F(x) = \int_1^x 2\left(1-\frac{1}{y^2}\right)dy = 2\left(y+\frac{1}{y}\right)\Big]_1^x = 2\left(x+\frac{1}{x}\right)-4$, so

$$F(x) = \begin{cases} 0 & x<1 \\ 2\left(x+\frac{1}{x}\right)-4 & 1 \le x \le 2 \\ 1 & x>2 \end{cases}$$

b. $2\left(x_p+\frac{1}{x_p}\right)-4 = p \Rightarrow 2x_p^2-(4-p)x_p+2=0 \Rightarrow x_p = \frac{1}{4}[4+p+\sqrt{p^2+8p}]$ To find $\tilde{\mu}$, set p = .5 $\Rightarrow \tilde{\mu} = 1.64$

c. $E(X) = \int_1^2 x \cdot 2\left(1-\frac{1}{x^2}\right)dx = 2\int_1^2\left(x-\frac{1}{x}\right)dx = 2\left(\frac{x^2}{2}-\ln(x)\right)\Big]_1^2 = 1.614$

$E(X^2) = 2\int_1^2 (x^2-1)dx = 2\left(\frac{x^3}{3}-x\right)\Big]_1^2 = \frac{8}{3} \Rightarrow$ Var(X) = .0626

d. Amount left = max(1.5 – X, 0), so

E(amount left) = $\int_1^2 \max(1.5-x,0)f(x)dx = 2\int_1^{1.5}(1.5-x)\left(1-\frac{1}{x^2}\right)dx = .061$

4. It has been reported that approximately 60% of U.S. households have two or more television sets. Suppose that $n = 15$ U.S. households are sampled and x is the number of households that have two or more television sets.

a. [1] What is the probability distribution for x?

b. [1] Find $P(x \le 8)$.

c. [1] What is the probability that x exceeds eight?

d. [1] What is the largest value of c for which $P(x \le c) \le 0.10$?

Sol:

a. x has a binomial distribution with $n = 15$ and $p = 0.6$, i.e.,

$$p(x) = \binom{15}{x}(0.6)^x(0.4)^{15-x}$$

b. $P(x \le 8) = 0.390$

c. $P(x > 8) = 1 - P(x \le 8) = 1 - 0.390 = 0.61$

d. $P(x \le c) = 0.1$, by using table c is at most 6.

5. Suppose that 50% of all young adults prefer McDonald's to Burger King when asked to state a preference. A group of 100 young adults were randomly selected and their preference recorded.

   a. [0.5] What is the probability that more than 60 preferred McDonald's?

   b. [0.5] What is the probability that between 40 to 60 (inclusive) preferred McDonald's?

   c. [0.5] What is the probability that between 40 to 60 (inclusive) preferred Burger King?

   Sol:

   Define x to be the number of young adults who prefer McDonald's. Then x has a binomial distribution with $n = 100$ and $p = .5$. Use the **Calculating Binomial Probabilities** applet.

   a $\quad P(61 \le x \le 100) = .0176$

   b $\quad P(40 \le x \le 60) = .9648$

   c $\quad$ If 40 prefer Burger King, then 60 prefer McDonalds, and vice versa. The probability is the same as that calculated in part b, since $p = .5$.

6. A particular industrial product is shipped in lots of 20 to customers. Testing whether an item is defective is costly, hence the manufacturer select a random sample of 4 items from each lot and rejects the lot for shipment if any of these items are defective.

   a. [0.5] What is the probability of accepting a lot for shipment if it contains 4 defective items?

   b. [0.5] What is the probability of accepting a lot for shipment if it contains 1 defective items?

   c. [0.5] What is the probability of rejecting a lot for shipment if it doesn't contain any defective item?

   Sol:

   Let random variable X = no. of defective in sample of 4. X is hypergeometric random variable with $N = 20,\ n = 4$ [2]

   a) $M = 4 \to p(x) = \frac{\binom{4}{x}\binom{16}{4-x}}{\binom{20}{4}}, \quad x = 0, 1, 2, 3, 4$

   $P(\text{accept shipment}) = P(X = 0) = \frac{\binom{4}{0}\binom{16}{4}}{\binom{20}{4}} = 0.3756$

   b) Here $M = 1 \to p(x) = \frac{\binom{1}{x}\binom{19}{4-x}}{\binom{20}{x}}, \quad x = 0, 1$

   $P(\text{accept shipment}) = P(X = 0) = \frac{\binom{1}{0}\binom{19}{4}}{\binom{20}{4}} = 0.8$

   c) P(reject)=0, since no defective in shipment, there can't be any in sample

7. For each of the following determine the value of c for which $p(x)$ is a pmf:

a) [1] $p(x) = c(1/6)^x, \qquad x = 1, 2, 3, \cdots$
b) [1] $p(x) = cx, \qquad x = 1, 2, 3, \cdots, n$
Sol:
The value of constant c is found from the condition $\sum_{\text{all } x} p(x) = 1$
a) Using the geometric formula

$$\sum_{k=0}^{\infty} q^k = \frac{1}{1-q}\ , \ |q| < 1,$$

we get

$$1 = c\sum_{x=1}^{\infty}(1/6)^x = c\left(\sum_{x=0}^{\infty}(1/6)^x - 1\right) = c\left(\frac{1}{1-1/6} - 1\right) = \frac{c}{5}$$

hence $c = 5$.
b) Recall that

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

therefore

$$c\sum_{x=1}^{n}\frac{n(n+1)}{2} = 1$$

and hence $c = \frac{2}{n(n+1)}$.

8. Suppose that earthquakes occur in a certain region of California, in accordance with a Poisson process, at a rate of five per year.
a) [1] What is the probability that a given eight-week period has at most two earthquakes? (note: Take 1 year=53 weeks.)
b.) [1] What is the probability of no earthquakes in two years?
c.) [2] What is the probability that in exactly two of the next eight years no earthquakes will occur?
Sol:
The rate of earthquakes is 5 per year
a) Let X be the number of earthquakes during an 8-week period. Then X has Poisson distribution with parameter 5(8/52)=10/13 and the probability is

$$P(X \leqslant 2) = e^{-10/13}[(10/13)^0/0! + (10/13)^1/1! + (10/13)^2/2!] = 0.957$$

b) Let Y be the number of earthquakes during a 2-year period. The probability is then

$$P(Y = 0) = e^{-10} = 0.000045$$

c) Let p be probability of no earthquake in one year. Then, with N being number of earthquakes per 1-year period,

$$p = P(N = 0) = e^{-5} = 0.0067.$$

Suppose that a year is called a success if during its course no earthquakes occur. Of the next eight years, let Z be the number of years in which no earthquakes will occur. Then Z is a binomial random variable with parameters 8 and p. Thus

$$P(Z = 2) = \binom{8}{2} p^2 (1 - p)^6 = \binom{8}{2} (0.0067)^2 (1 - 0.0067)^6 = 0.0012$$