27/30

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University of Ottawa

Department of Mathematics and Statistics

MAT1302D : Mathematical Methods II
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Second Midterm Exam - Version B
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Student # James in Biostatistics

Instructions:

1. You have 75 minutes to complete this exam.
2. All work to be considered for grading should be written in the space provided. Show complete work. The reverse side of pages is for scrap work. If you find that you need extra space in order to answer a particular question, you should continue on the reverse side of the page and indicate this **clearly**. Otherwise, the work written on the reverse side of pages will not be considered for marks.
3. Write your name at the top of each page.
4. No notes, books, scrap paper, calculators or other electronic devices are allowed.
5. You may use the last page of the exam as scrap paper.
6. Cellular phones, unauthorized electronic devices or course notes (unless an open-book exam) are not allowed during this exam. **Phones and devices must be turned off and put away in your bag**. Do not keep them in your possession, such as in your pockets. If caught with such a device or document, the following may occur: you will be asked to leave immediately the exam and academic fraud allegations will be filed which may result in you obtaining a 0 (zero) for the exam.

By signing below, you acknowledge that you have ensured that you are complying with the above statement.

Signatu

Please do not write in the table below.

Question	1	2	3	4	5	6	Total
Maximum	6	4	5	5	5	5	30
Grade							

1) [6 points] For each of the following statements, determine if it is **true (T)** or **false (F)**. (Only mark T or F, no other work required.)

[For each **correct answer**, you will receive **1 point**. For each **incorrect answer**, you will **lose additional 0.5 points**; but your total score for this problem cannot be negative. You have the option to leave any problem blank.]

✓ [F] (i) Every square matrix has an inverse.

✓ [T] (ii) If the matrix A has an inverse, then the transpose A^t has an inverse.

✓ [F] (iii) If A and B are invertible $n \times n$ matrices, then the sum $A + B$ is invertible.

✓ [T] (iv) If A and B are invertible $n \times n$ matrices, then the product AB is invertible.

✓ [F] (v) If A is a square matrix given as $A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$, then $A^2 = \begin{bmatrix} 9 & 1 \\ 4 & 16 \end{bmatrix}$.

✓ [T] (vi) A homogeneous linear system is always consistent.

6/6

$\begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$

$3 \cdot 3 + 1 \cdot 2$ $\quad$ $3 \cdot 1 + 1 \cdot 4$

11

$$\begin{array}{ccc} 2 & 1 & 2 \\ 1 & -1 & 1 \\ -1 & 2 & 0 \end{array}$$

2) [4 points] Is the matrix $A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & -1 & 0 \\ -2 & 1 & 0 \end{bmatrix}$ invertible? If it is invertible, find its inverse. Otherwise, shortly explain why it has no inverse. $2(1) = 2+1 = 3$

$$A^{-1} = \left[\begin{array}{ccc|ccc} 2 & 1 & -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ -2 & 1 & 0 & 0 & 0 & 1 \end{array}\right] R_1+R_3 \to R_3 \quad \left[\begin{array}{ccc|ccc} 2 & 1 & -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 2 & -1 & 1 & 0 & 1 \end{array}\right] -2R_2+R_1 \to R_1 \quad \left[\begin{array}{ccc|ccc} 0 & 3 & -1 & 1 & -2 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 2 & -1 & 1 & 0 & 1 \end{array}\right] R_1 \leftrightarrow R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 3 & -1 & 1 & -2 & 0 \\ 0 & 2 & -1 & 1 & 0 & 1 \end{array}\right] -R_3+R_2 \to R_2 \quad \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & -2 & -1 \\ 0 & 2 & -1 & 1 & 0 & 1 \end{array}\right] -2R_2+R_3 \to R_3 \quad \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & -2 & -1 \\ 0 & 0 & -1 & 1 & 4 & 3 \end{array}\right] R_2+R_1 \to R_1$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 & -2 & -1 \\ 0 & 0 & 1 & -1 & -4 & -3 \end{array}\right]$$

double check work
in second matrix!
look at the addition!

$$A^{-1} = \begin{bmatrix} 0 & -1 & -1 \\ 0 & -2 & -1 \\ -1 & -4 & -3 \end{bmatrix}$$

4/4

$a \ b$
$c \ d$

$1 \ 2$
$3 \ 0$

$ad - bc \neq 0$ $\quad$ $\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$\frac{1}{ad-bc}\begin{bmatrix} \cancel{a} & -\cancel{b} \\ \cancel{c} & \cancel{d} \end{bmatrix}$ $\quad$ $a=1 \quad b=7 \quad c=-3 \quad d=-1$

3) [5 points] Let $A = \begin{bmatrix} 1 & -3 \\ 2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 7 \\ -3 & -1 \end{bmatrix}$.

Verify that B^{-1} exists, and then solve the equation $B(X^t + A)B^{-1} = 3I_2$ for X. (Here I_2 stands for the 2×2 identity matrix.) Indicate each step clearly.

1

$$B^{-1} = \frac{1}{(1)(-1) - (7)(-3)} \begin{bmatrix} -1 & -7 \\ 3 & 1 \end{bmatrix}$$

$$B^{-1} = \frac{1}{-1-(-21)} \begin{bmatrix} -1 & -7 \\ 3 & 1 \end{bmatrix}$$

$$B^{-1} = \frac{1}{20} \begin{bmatrix} -1 & -7 \\ 3 & 1 \end{bmatrix}$$

$$B^{-1} = \begin{bmatrix} -\frac{1}{20} & -\frac{7}{20} \\ \frac{3}{20} & \frac{1}{20} \end{bmatrix}$$

2

$$B(X^t + A)B^{-1} = 3I_2$$

$$B(X^t + A)B^{-1}B = 3I_2B$$ ← multiplied B on the right on bothe sides of equal sign

$$B(X^t + A)I_2 = 3I_2B$$

$$B^{-1}B(X^t + A) = B^{-1}3I_2B$$ ← multiplied B^{-1} on the left on both sides of equal sign

$$I_2(X^t + A) = B^{-1}3I_2B$$

$$X^t + A = B^{-1}3I_2B$$

$$X^t = B^{-1}3I_2B - A$$

$$(X^t)^t = (B^{-1}3I_2B - A)^t$$ ← multiplied the transpose on each side

$$X = (B^{-1}3I_2B - A)^t$$

$AB \neq BA$

4/5

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

d -b

-c a

ad - bc

a = 1 b = -3

c = 2 d = -2

1(-2) - (-3)(2)

-2 + 6

4) [5 points] Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -3 \\ 2 & -2 \end{bmatrix}$.

Find A^{-1}, B^{-1}, A^t, B^t, AB, and $(AB)^{-1}$. Verify that $(AB)^{-1} \neq A^{-1}B^{-1}$.
$\boxed{\text{Also verify that } (A+B)^t = A^t + B^t.}$ → DONE on BLANK page

$A^t = \begin{bmatrix} 0 & 1 \\ 1 & 4 \end{bmatrix}$ $B^t = \begin{bmatrix} 1 & 2 \\ -3 & -2 \end{bmatrix}$

$A^{-1} = \left[\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ 1 & 4 & 0 & 1 \end{array}\right]$ $R_1 \leftrightarrow R_2$

$A^{-1} = \left[\begin{array}{cc|cc} 1 & 4 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right]$ $-4R_2 + R_1 \to R_1$

$A^{-1} = \left[\begin{array}{cc|cc} 1 & 0 & -4 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right]$

$A^{-1} = \begin{bmatrix} -4 & 1 \\ 1 & 0 \end{bmatrix}$

$\boxed{B^{-1} \to \text{This done on the blank page} ✓}$

3/5

$\begin{bmatrix} 0 & 1 \\ 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & -3 \\ 2 & -2 \end{bmatrix}$

$A^{-1} \begin{bmatrix} -4 & 1 \\ 1 & 0 \end{bmatrix} \cdot B^{-1} \begin{bmatrix} -6 & 9 \\ -6 & 3 \end{bmatrix}$

$= \begin{matrix} 0 \cdot 1 + 1 \cdot 2 & 0(-3) + 1(-2) \\ 1(1) + (4)(2) & 1(-3) + (4)(-2) \end{matrix}$

$\begin{matrix} -24 & -6 & 36 \\ 4(-6) + (1)(-6) & (4)(9) + (1)(3) \\ (1)(-6) + (0)(-6) & (1)(9) + (0)(3) \\ -6 & & \end{matrix}$

$1\begin{bmatrix} 0 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 4 \end{bmatrix}$

$-3\begin{bmatrix} 0 \\ 1 \end{bmatrix} + (-2)\begin{bmatrix} 1 \\ 4 \end{bmatrix}$

$(AB)^{-1} = \cancel{\begin{bmatrix} 2 & -2 \\ 9 & -11 \end{bmatrix}} \neq A^{-1}B^{-1} = \cancel{\begin{bmatrix} -30 & 39 \\ -6 & 9 \end{bmatrix}}$

$\therefore \quad (AB)^{-1} \neq A^{-1}B^{-1}$

$\begin{matrix} 0 \\ 1 \end{matrix} + \begin{matrix} 2 \\ 4 \end{matrix} \to \begin{matrix} 2 \\ 5 \end{matrix} \quad \begin{matrix} -2 \\ -11 \end{matrix}$

5) (a) [4 points] Find the general solution of the following linear homogeneous system of equations. Write the answer in **vector parametric form**.

$$\begin{aligned} -x_1 &+ x_2 & &= 0 \\ 2x_1 &- x_2 &+ 2x_3 &= 0 \\ & x_2 &+ 2x_3 &= 0 \end{aligned}$$

$$\left[\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 2 & -1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \xrightarrow{2R_1+R_2 \to R_1} \left[\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \xrightarrow{-R_1} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \xrightarrow{R_2+R_1 \to R} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right]$$

$$\xrightarrow{-R_2+R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 \quad x_2 \quad x_3$

$x_1 = 0$

$x_2 = -2x_3$

$x_3 = \text{free}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$$

4/4

(b) [1 point] The values $x_1 = 1, x_2 = 4$, and $x_3 = 1$ form a solution for the linear non-homogeneous system given below.

$$\begin{aligned} -x_1 &+ x_2 & &= 3 \\ 2x_1 &- x_2 &+ 2x_3 &= 0 \\ & x_2 &+ 2x_3 &= 6 \end{aligned}$$

$v = pv + h$ ✓ or something like that

Without using row reduction, find the general solution for the above linear non-homogeneous system. Write your answer in vector parametric form.

(Hint: Note that the coefficient matrix of the linear system in part (b) is the same as the one in part (a).)

1/1

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$$

~~$-1+4=3$~~

$-(1) + (4) = 3$

$2(1) - (4) + 2(1) = 0$

$(4) + 2(1) = 6$

$x_3 = 1$

$V =$

T
A

6) An economy is based on two sectors: Tourism and Agriculture. In order to produce 1 unit of output, the Tourism sector uses 0.4 units from the Tourism and 0.5 units from the Agriculture sector. Further, to produce 1 unit of output, the Agriculture sector uses 0.3 units from Tourism and 0.7 units from the Agriculture sector.

(a) [1 point] Write down the consumption matrix C for this economy.

$$C = \begin{bmatrix} 0.4 & 0.3 \\ 0.5 & 0.7 \end{bmatrix}$$

1/1

(b) [1 point] Write down Leontief's input-output equation.

$$x - Cx = d$$
$$\downarrow$$
$$(I - C)x = d$$

1/1

(c) [3 points] Determine the production levels required to meet a final demand of 6 units from the Tourism sector and 8 units from the Agriculture sector.

Find $\underline{\underline{X}}$

$$d = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$$

$$(I - C)x = d$$
$$(I-C)^{-1}(I-C)x = (I-C)^{-1}d$$
$$x = (I-C)^{-1}d$$

① $\left(\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} - \begin{bmatrix} 0.4 & 0.3 \\ 0.5 & 0.7 \end{bmatrix}\right) x = d$

$$\begin{pmatrix} 0.6 & -0.3 \\ -0.5 & 0.3 \end{pmatrix} x = d$$

② $\left[\begin{array}{cc|cc} 0.6 & -0.3 & 1 & 0 \\ -0.5 & 0.3 & 0 & 1 \end{array}\right]$ $\begin{matrix} 10R_1 \\ 10R_2 \end{matrix}$

$\left[\begin{array}{cc|cc} 6 & -3 & 10 & 0 \\ -5 & 3 & 0 & 10 \end{array}\right]$ $R_2 + R_1 \to R_1$

$$\downarrow$$

$\left[\begin{array}{cc|cc} 1 & 0 & 10 & 10 \\ -5 & 3 & 0 & 10 \end{array}\right]$ $5R_1 + R_2 \to R_2$

$\left[\begin{array}{cc|cc} 1 & 0 & 10 & 10 \\ 0 & 3 & 0 & 10 \end{array}\right]$ $\frac{1}{3}R_2$

$\left[\begin{array}{cc|cc} 1 & 0 & 10 & 10 \\ 0 & 1 & 0 & \frac{10}{3} \end{array}\right]$

③ $x = (I - C)^{-1} d$

$$x = \begin{bmatrix} 10 & 10 \\ 0 & \frac{10}{3} \end{bmatrix} \begin{bmatrix} 6 \\ 8 \end{bmatrix}$$

$$x = 6\begin{bmatrix} 10 \\ 0 \end{bmatrix} + 8\begin{bmatrix} 10 \\ \frac{10}{3} \end{bmatrix}$$

$$x = \begin{matrix} 60 \\ 0 \end{matrix} + \begin{matrix} 80 \\ \frac{80}{3} \end{matrix}$$

$$x = \begin{bmatrix} 140 \\ \frac{80}{3} \end{bmatrix}$$

3/3