MAT 1332, Winter 2017, Assignment 6

Due MONDAY April 10 in the math department dropboxes by 7:00pm.

Late assignments will not be accepted; nor will unstapled assignments.

Professors in the math department will not lend you a stapler; do not ask for one.

Please print double sided to save paper.

Instructor (circle one): Guy Beaulieu Robert Smith? Xiaoying Wang

DGD (circle one): 1 2 3 4

Name (Prime student) ____________________ Student Number ____________________

Student Name ____________________ Student Number ____________________

Student Name ____________________ Student Number ____________________

By signing below, we declare that this work is our own, that we have not copied from any other individual or other source and that all students contributed equally.

Signatures ____

QUESTION 1. Which of the following is NOT an entry in the Jacobian of

$$F(x,y) = \begin{bmatrix} x^2e^y + 2x\sin(x^y) \\ x^4\sin(\ln(x^2)) - 3ye^{-2x} \end{bmatrix}$$

at the point $(1,0)$? Circle all that apply. (Marks will be deducted for incorrect answers.)

A) 2 B) 6 C) 1 D) $2+2\sin 1$ E) $1+2\cos 1$ F) -3

The Jacobian of $F = \begin{pmatrix} f \\ g \end{pmatrix}$ is given by

$$J = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}.$$

We thus have

$$\frac{\partial f}{\partial x} = 2xe^y + 2\sin(x^y) + 2x\cos(x^y)yx^{y-1} \qquad \frac{\partial f}{\partial y} = x^2 + 2x\cos(x^y)x^y\ln x$$

$$\frac{\partial g}{\partial x} = 4x^3\sin(\ln(x^2)) + x^4\cos(\ln(x^2))\frac{2x}{x^2} + 6ye^{-2x} \qquad \frac{\partial g}{\partial y} = -3e^{-2x}$$

At the point $(1,0)$, we have

$$\frac{\partial f}{\partial x} = 2+2\sin 1 \qquad \frac{\partial f}{\partial y} = 1$$

$$\frac{\partial g}{\partial x} = 2 \qquad \frac{\partial g}{\partial y} = -3e^{-2}$$

Thus the answer is B and E.

(1 mark for B and 1 mark for E and 1 mark for F)

QUESTION 2. Consider the following system of linear differential equations:

$$\frac{dx}{dt} = -4x + 6y$$
$$\frac{dy}{dt} = x - 3y$$

(a) Find the eigenvalues and eigenvectors associated with the system.

The matrix associated to the system is the matrix of coefficients in the differential equations:

$$A = \begin{pmatrix} -4 & 6 \\ 1 & -3 \end{pmatrix}.$$

Its eigenvalues are the roots of

$$\det(A - \lambda I) = \det \begin{pmatrix} -4-\lambda & 6 \\ 1 & -3-\lambda \end{pmatrix} = (-4-\lambda)(-3-\lambda) - 6 = \lambda^2 + 7\lambda + 6 = (\lambda+6)(\lambda+1),$$

so $\lambda = -6, -1$. (We could use the quadratic formula instead of factoring directly; in general the quadratic formula will be necessary.)

We find the eigenvectors for each eigenvalue λ by solving $(A - \lambda I)\mathbf{v} = \mathbf{0}$ by row reduction. For $\lambda = -6$,

$$A - \lambda I = \begin{pmatrix} 2 & 6 \\ 1 & 3 \end{pmatrix},$$

and the augmented matrix is row-reduced to

$$\left(\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right).$$

The general solution to this (i.e., to $x_1 + 3x_2 = 0$) is

$$\mathbf{v} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -3r \\ r \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \end{pmatrix} r, r \in \mathbb{R}, r \neq 0.$$

Similarly we find that

$$\mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} s, s \in \mathbb{R}, s \neq 0$$

are the eigenvectors for the eigenvalue $\lambda_2 = -1$.

(b) Write down the general solution formula for the system.

The general solution is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2 = C_1 e^{-6t}\begin{pmatrix} -3 \\ 1 \end{pmatrix} + C_2 e^{-t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -3C_1 e^{-6t} + 2C_2 e^{-t} \\ C_1 e^{-6t} + C_2 e^{-t} \end{pmatrix},$$

so $x(t) = -3C_1 e^{-6t} + 2C_2 e^{-t}$, $y(t) = C_1 e^{-6t} + C_2 e^{-t}$, with C_1 and C_2 arbitrary constants.

(c) Find $y(t)$ if $x(0) = 6, y(0) = 4$.

At $t = 0$, $x = -3C_1 + 2C_2$ and $y = C_1 + C_2$, so we need to solve the linear system

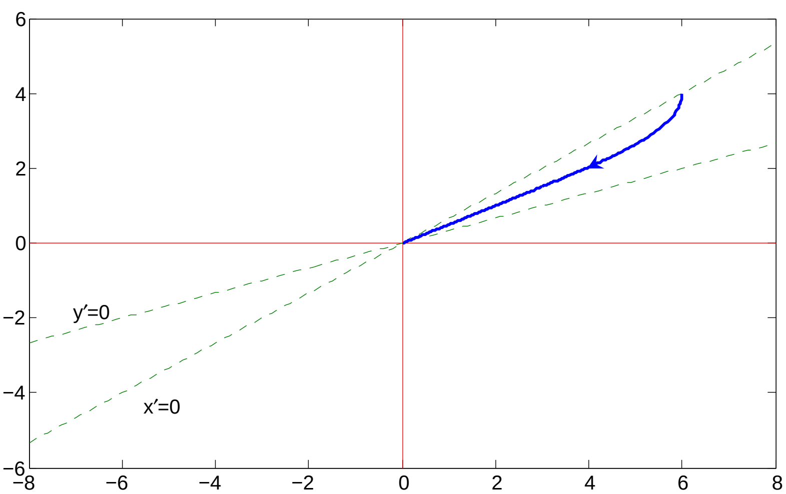
$$-3C_1 + 2C_2 = 6$$
$$C_1 + C_2 = 4$$

We can use any method we like, such as row-reduction, to find that there is a single solution, with $C_1 = 2/5$, $C_2 = 18/5$. The solution for $y(t)$ is therefore

$$y(t) = \frac{2}{5}e^{-6t} + \frac{18}{5}e^{-t}$$

(d) Draw the x- and y-nullclines. Carefully sketch the solution curve for the initial condition in part (c) in the phase plane on the same graph.

[3] The x-nullcline is the solutions to $-4x + 6y = 0$, which is the line $y = \frac{2}{3}x$. The y-nullcline is the solutions to $x - 3y = 0$, which is the line $y = \frac{1}{3}x$.



Note that the initial condition starts on the x-nullcline, but solutions DO NOT travel down the nullcline. (They can't, as it's a vertical nullcline.) They can't cross the y-nullcine either. Since the equilibrium is a sink, they have to travel between the nullclines to the equilibrium.

(0.5 marks for each nullcline, 1 mark for initially heading vertically down, 1 for ending at the origin. Lose 1 mark if the solutions travel down the nullcline.)

(e) Is the point $(0, 0)$ stable or unstable? Classify this equilibrium.

Note that $(0, 0)$ is the only equilibrium point. This will be true for any **linear** system of differential equations. Since both of the eigenvalues (-5 and -1) have negative real part, this equilibrium is stable. This equilibrium is a **sink**.

We can further see that the system is stable since the general solution in part (b) heads towards $(0, 0)$ as time t gets larger and larger, regardless of the initial conditions.

QUESTION 3. Consider the following system of linear differential equations:

$$\frac{dx}{dt} = -16x + 42y$$
$$\frac{dy}{dt} = 7x - 9y$$

Draw the x- and y-nullclines. On the same graph, sketch the solution with initial condition $x(0) = 12$, $y(0) = -4$.

[2] We have

$$A = \begin{bmatrix} -16 & 42 \\ 7 & -9 \end{bmatrix}$$
$$\det(A - \lambda I) = \begin{bmatrix} -16-\lambda & 42 \\ 7 & -9-\lambda \end{bmatrix}$$
$$= (16+\lambda)(9+\lambda) - 294$$
$$= (\lambda - 5)(\lambda + 30)$$

Hence the equilibrium is a saddle.

For $\lambda = 5$, we have

$$\left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

so an eigenvector is $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

For $\lambda = -30$, we have

$$\left[\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

so an eigenvector is $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$.

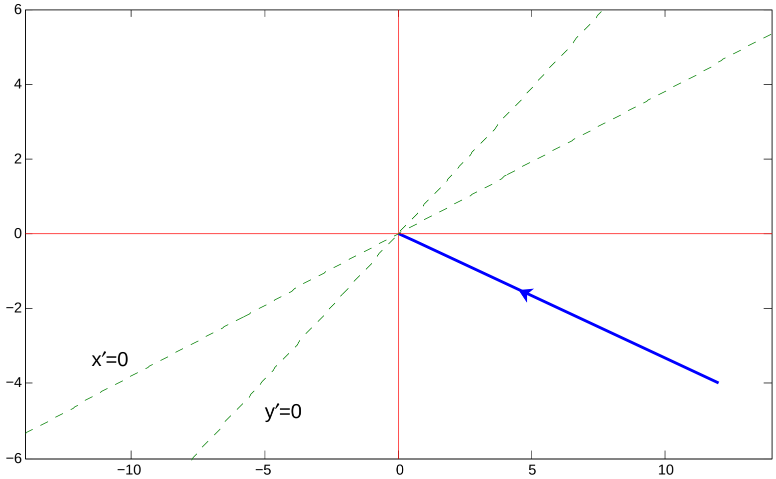
Hence the general solution is

$$\begin{bmatrix} x \\ y \end{bmatrix} = C_1 e^{5t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 e^{-30t} \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

For the initial conditions, $C_1 = 0$ and $C_2 = -4$, so the particular solution is

$$\begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} = -4e^{-30t} \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

As $t \to \infty$, $\tilde{x} \to 0$ and $\tilde{y} \to 0$. It follows that solutions starting at this initial condition go to the equilibrium at $(0, 0)$. (Note that this isn't true for most initial conditions in this system, since a saddle is unstable, but it is true for a few, because of the nature of saddle.)



(1 mark for heading in the right direction initially, 1 for ending at the origin)

QUESTION 4. There is a war for survival between werewolves (L) and vampires (V) that thankfully leaves humans unaffected (phew!). Since a vampire can infect a werewolf and turn it into vampire, and vice versa, the population of these two monstrous species is modelled by the following system:

$$\frac{dL}{dt} = 100L - 5VL$$
$$\frac{dV}{dt} = V(100 - V) - VL$$

(a) Find all biologically meaningful equilibrium points.
From the first equation, either $L = 0$ or $V = 20$.
Plugging $L = 0$ into the second equation gives either $V = 0$ or $V = 100$.
Plugging $V = 20$ into the second equation gives $L = 80$. Thus the equilibria are

$$(L, V) = (0, 0), (0, 100), (80, 20).$$

All are biologically meaningful.

(b) Show that the Jacobian is given by

$$J(L, V) = \begin{bmatrix} 100 - 5V & -5L \\ -V & 100 - 2V - L \end{bmatrix}$$

Obvious.

(c) For each of the biologically meaningful equilibria from (a), find the eigenvalues of the Jacobian matrix.

We have

$$J(0,0) = \begin{bmatrix} 100 & 0 \\ 0 & 100 \end{bmatrix}$$

so the eigenvalues are $\lambda = 100, 100$.

$$J(0,100) = \begin{bmatrix} -400 & 0 \\ -100 & -100 \end{bmatrix}$$

so the eigenvalues are $\lambda = -400, -100$.

$$\begin{aligned} J(80,20) &= \begin{bmatrix} 0 & -400 \\ -20 & -20 \end{bmatrix} \\ \det(J(80,20) - \lambda I) &= \begin{bmatrix} -\lambda & -400 \\ -20 & -20-\lambda \end{bmatrix} \\ &= \lambda^2 + 20\lambda - 8000 \\ \lambda &= \frac{-20 \pm \sqrt{32400}}{2} = \frac{-20 \pm 180}{2} = 80, -100. \end{aligned}$$

(d) Classify each equilibrium and determine its stability.

$(0,0)$ is a source and is unstable.

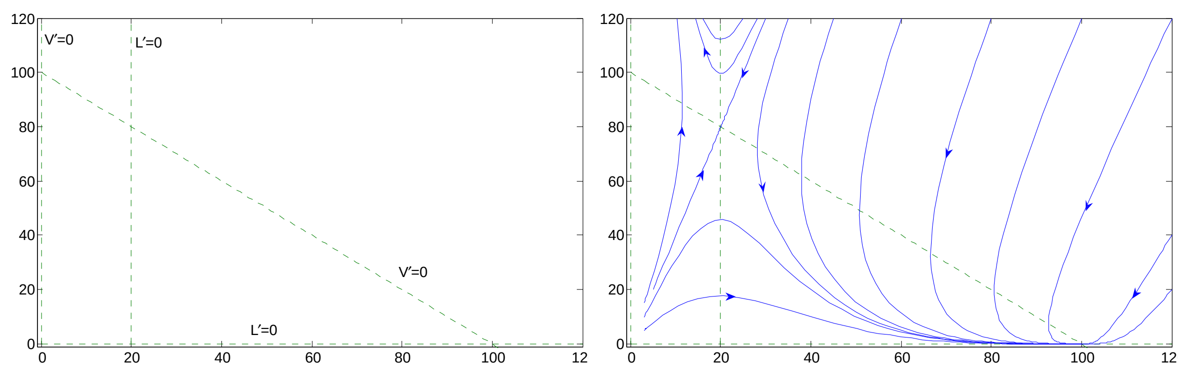
$(0,100)$ is a sink and is stable.

$(80,20)$ is a saddle and is unstable.

(e) Draw the nullclines in the region $L \geq 0, V \geq 0$. On the same graph, sketch several solutions that start in the interior (i.e., not on the axes).

The L-nullcines are $L = 0$ and $V = 20$.

The V-nullcines are $V = 0$ and $L = 100 - V$.



(g) If there are 20 vampires and 5 werewolves initially, what happens eventually?

Since $L(0) < 80$ and the initial conditions are not on an equilibrium, solutions will head towards the stable sink at $(100, 0)$. That is, eventually there will be 100 vampires and no werewolves.

QUESTION 5. Consider a disease that propagates according to the system

$$\frac{dx}{dt} = 20 - 0.4xy - 0.8x$$
$$\frac{dy}{dt} = 0.2xy - 14y$$

where x represents susceptible individuals, y represents infected individuals.

(a) Find all biologically meaningful steady states.

Biologically meaningful here simply means that the numbers are not negative. The steady states (= equilibrium points) are the places where both $20-0.4xy-0.8x = 0$ and $0.2xy-14y = 0$. The second equation is easier (since we can factor it) so we deal with it first: $y(0.2x-14) = 0$ when $y = 0$ or when $x = 14/0.2 = 70$. For each of these cases we plug the given value into the first equation (which must also hold).

[1] If $y = 0$, then the first equation says that $20 - 0.8x = 0$, so $x = 20/0.8 = 25$. Therefore $(25, 0)$ is one equilibrium.

[1] The only other case is when $x = 70$. Here, the first equation says that $20 - 0.4(70)y - 0.8(70) = 0$, so $28y = -36$ and $y = -1.28$. Therefore $(70, -1.28)$ is another equilibrium, and there are no others. This equilibrium point is not biologically meaningful since its second coordinate is negative. **(1 point for ruling out this equilibrium with an explanation)**

(b) Show that the Jacobian matrix of this system is given by

$$\begin{bmatrix} -0.8 - 0.4y & -0.4x \\ 0.2y & 0.2x - 14 \end{bmatrix}$$

The Jacobian of

$$\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} 20 - 0.4xy - 0.8x \\ 0.2xy - 14y \end{pmatrix}$$

is given by

$$J = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}.$$

We just have to confirm four partial derivatives were given correctly. So, for example,

$$\frac{d}{\partial x}(20 - 0.4xy - 0.8x) = -0.8 - 0.4y.$$

(c) For the biologically meaningful steady states from (a), find the eigenvalues of the Jacobian matrix.

[2] We have one biologically meaningful steady state: (25,0). We plug $x = 25$, $y = 0$ into the formula given in part (b) for J:

$$J(25,0) = \begin{pmatrix} -0.8 & -10 \\ 0 & -9 \end{pmatrix}.$$

This matrix is upper-triangular (since the only entry below the main diagonal is zero), so its eigenvalues are its diagonal entries: -0.8 and -9. You should compute the characteristic polynomial to verify that these are in fact the eigenvalues.

(1 mark for each eigenvalue)

(d) Determine the stability and classify the biologically meaningful steady states.

[1] Since the eigenvalues of the Jacobian matrix at the equilibrium have negative real part (in fact, are negative real numbers), we can conclude that this equilibrium is stable. In fact, it is a stable sink.

What this means in concrete terms is that starting from any population with any infection rate, after enough time the end result will be that x is very close to 40 and y is very close to 0; in particular the disease will be wiped out in time.

(0.5 for stability, 0.5 for classification)