MAT 1332, Winter 2017, Assignment 4
Due Wednesday March 8 in the math department dropboxes by 7:00pm.
Late assignments will not be accepted; nor will unstapled assignments.
Professors in the math department will not lend you a stapler; do not ask for one.
Please print double sided to save paper.

Instructor (circle one): Guy Beaulieu   Robert Smith?   Xiaoying Wang

DGD (circle one): 1   2   3   4

Name (Prime student) ______________________ Student Number ____________________
Student Name ______________________________ Student Number ____________________
Student Name ______________________________ Student Number ____________________

By signing below, we declare that this work is our own, that we have not copied from any other individual or other source and that all students contributed equally.

Signatures ____

QUESTION 1. For the system of linear equations

$$\begin{array}{ccccccc} x & + & 4y & + & 16z & = & 4 \\ 2x & + & 9y & + & 39z & = & 3 \\ x & + & ay & + & a^2z & = & a \end{array}$$

a) determine the values of a for which the system has

[1] (i) no solution $a = 3$

[1] (ii) infinitely many solutions $a = 4$

[1] (iii) a unique solution $a \notin \{3, 4\}$

(1 mark for each answer. No part marks.)

The augmented matrix of the system is

$$A = \left[\begin{array}{ccc|c} 1 & 4 & 16 & 4 \\ 2 & 9 & 39 & 3 \\ 1 & a & a^2 & a \end{array}\right]$$

We perform the following operations, where R_i is row i: $R_2 \rightsquigarrow R_2 - 2R_1$, $R_3 \rightsquigarrow R_3 - R_1$, $R_3 \rightsquigarrow R_3 - (a-3)R_2$, and obtain

$$A \sim \left[\begin{array}{ccc|c} 1 & 4 & 16 & 4 \\ 0 & 1 & 7 & -5 \\ 0 & a-4 & a^2-16 & a-4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 4 & 16 & 4 \\ 0 & 1 & 7 & -5 \\ 0 & 0 & a^2-16-7a+28 & 6(a-3) \end{array}\right].$$

Since $a^2-16-7a+28=(a-3)(a-4)$ we get:

$$\left[\begin{array}{ccc|c} 1 & 4 & 16 & 4 \\ 0 & 1 & 7 & -5 \\ 0 & 0 & (a-3)(a-4) & 6(a-4) \end{array}\right]$$

- If $a=3$, then the last row of the matrix is is $\left[\begin{array}{ccc|c} 0 & 0 & 0 & -12 \end{array}\right]$. Hence the system is inconsistent.
- If $a=4$ then

$$M=\left[\begin{array}{ccc|c} 1 & 4 & 16 & 4 \\ 0 & 1 & 7 & -5 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -12 & 24 \\ 0 & 1 & 7 & -5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

  Hence the system has infinitely many solutions.
- If $a \notin \{3,4\}$, then $(a-3)(a-4) \neq 0$ and so the system is uniquely solvable.

b) In case (ii) above describe all solutions.

The RREF of the matrix is

$$\left[\begin{array}{ccc|c} 1 & 0 & -12 & 24 \\ 0 & 1 & 7 & -5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The corresponding linear system is

$$\begin{array}{ccccccc} x & & & - & 12z & = & 24 \\ & & y & + & 7z & = & -5 \end{array}$$

Thus z is a free variable. Putting $z=t$, the general solution is

$$(24+12t, -5-7t, t) \quad \text{(where } t \text{ is a free parameter)}$$

c) Consider the matrix

$$A=\begin{bmatrix} 1 & 4 & 16 \\ 2 & 9 & 39 \\ 1 & a & a^2 \end{bmatrix}$$

with $a=1$. Which of the following are NOT part of A^{-1}? Circle all that apply. (Marks will be deducted for incorrect answers.)

A) 2 B) 5 C) $\frac{7}{6}$ D) $\frac{1}{6}$ E) $\frac{37}{6}$

(1 mark for each correct answer. Subtract 1 mark for each incorrect answer. No part marks.)

[2] We set up the 3×6 matrix for the inverse.

$$\left[\begin{array}{ccc|ccc} 1 & 4 & 16 & 1 & 0 & 0 \\ 2 & 9 & 39 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right].$$

We use Gauss–Jordan elimination on this matrix $R_2 \rightsquigarrow R_2 - 2R_1, R_3 \rightsquigarrow R_3 - R_1$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 9 & 1 & 0 & 0 \\ 0 & 1 & 7 & -2 & 1 & 0 \\ 0 & -3 & -15 & -1 & 0 & 1 \end{array}\right].$$

Next: $R_3 \rightsquigarrow R_3 + 3R_2$ and then $R_3 \rightsquigarrow R_3/6$: $\left[\begin{array}{ccc|ccc} 1 & 4 & 16 & 1 & 0 & 0 \\ 0 & 1 & 7 & -2 & 1 & 0 \\ 0 & 0 & 1 & -\frac{7}{6} & \frac{1}{2} & \frac{1}{6} \end{array}\right]$. The next step is $R_1 \rightsquigarrow R_1 - 4R_2$ $\left[\begin{array}{ccc|ccc} 1 & 0 & -12 & 9 & -4 & 0 \\ 0 & 1 & 7 & -2 & 1 & 0 \\ 0 & 0 & 1 & -\frac{7}{6} & \frac{1}{2} & \frac{1}{6} \end{array}\right]$ and finally $R_1 \rightsquigarrow R_1 + 12R_3, R_2 \rightsquigarrow R_2 - 7R_3$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -5 & 2 & 2 \\ 0 & 1 & 0 & \frac{37}{6} & -\frac{5}{6} & -\frac{7}{6} \\ 0 & 0 & 1 & -\frac{7}{6} & \frac{1}{2} & \frac{1}{6} \end{array}\right]$. So the inverse is $\begin{bmatrix} -5 & 2 & 2 \\ \frac{37}{6} & -\frac{5}{6} & -\frac{7}{6} \\ -\frac{7}{6} & \frac{1}{2} & \frac{1}{6} \end{bmatrix}$. Hence the answers are B and C.

QUESTION 2. Given the following matrices and vectors

$$A = \begin{bmatrix} 1 & -11 & -9 \\ 3 & 0 & 10 \\ 14 & -2 & 0 \end{bmatrix}, B = \begin{bmatrix} 2 & -7 \\ 8 & 5 \\ 0 & 1 \end{bmatrix}, \vec{u} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \vec{v} = \begin{bmatrix} 1 \\ 5/3 \\ 0 \end{bmatrix}, \vec{w} = \begin{bmatrix} 0 \\ 1/3 \\ 3/5 \end{bmatrix}.$$

Compute the following if possible. If not possible, explain in one sentence why.

a) $A^T\vec{v} + 2B^T\vec{u}$.

[1] $A^T\vec{v} + 2B^T\vec{u}$ is not defined since $B^T\vec{u}$ is 2×1 and $A^T\vec{v}$ is 3×1.

b) $\vec{w}\vec{v}^T$

[1] $$\vec{v}\vec{w}^T = \begin{bmatrix} 0 & 1/3 & 3/5 \\ 0 & 5/9 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

c) $\vec{v}^T\vec{w}$

$\vec{v}^T\vec{w} = 1 \times 0 + 5/3 \times 1/3 + 0 \times 3/5 = 5/9$.

d) $A\vec{u} + 2\vec{v}^T\vec{w}$

$A\vec{u} + 2\vec{v}^T\vec{w}$ is not defined since $A^T\vec{u}$ is a 3×1 vector and $\vec{v}^T\vec{w}$ is 1×1.

e) AB

[1] $$A^TB = \begin{bmatrix} 26 & 22 \\ -22 & 75 \\ 62 & 113 \end{bmatrix}.$$

f) $B\vec{u}$

$B\vec{u}$ is not defined since B has 2 columns and $\vec{u}$ has 3 rows.

g) BA

[1] BA is not defined since B has 2 columns and A has 3 rows.

h) A^2

$$A^2 = \begin{bmatrix} -158 & 7 & -119 \\ 143 & -53 & -27 \\ 8 & -154 & -146 \end{bmatrix}.$$

i) B^2

[1] B^2 is not defined since it is not square. (The first B is 3×2, and the second is 3×2.)

QUESTION 3. Determine the matrix A such that:

$$\left(4A^T - \begin{bmatrix} 1 & 4 & -3 \\ -2 & 5 & 1 \end{bmatrix}\right)^T = \begin{bmatrix} -5 & 2 \\ 9 & 11 \\ -3 & 7 \end{bmatrix} + 4\begin{bmatrix} 3 & 5 & -4 \\ 7 & 13 & 2 \end{bmatrix}^T.$$

We calculate the both sides of the equation

$$\left(4A^T - \begin{bmatrix} 1 & 4 & -3 \\ -2 & 5 & 1 \end{bmatrix}\right)^T = 4(A^T)^T - \begin{bmatrix} 1 & 4 & -3 \\ -2 & 5 & 1 \end{bmatrix}^T = 4A - \begin{bmatrix} 1 & -2 \\ 4 & 5 \\ -3 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -5 & 2 \\ 9 & 11 \\ -3 & 7 \end{bmatrix} + 4\begin{bmatrix} 3 & 5 & -4 \\ 7 & 13 & 2 \end{bmatrix}^T = \begin{bmatrix} -5 & 2 \\ 9 & 11 \\ -3 & 7 \end{bmatrix} + 4\begin{bmatrix} 3 & 7 \\ 5 & 13 \\ -4 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & 2 \\ 9 & 11 \\ -3 & 7 \end{bmatrix} + \begin{bmatrix} 12 & 28 \\ 20 & 52 \\ -16 & 8 \end{bmatrix} = \begin{bmatrix} 7 & 30 \\ 29 & 63 \\ -19 & 15 \end{bmatrix}$$

Hence

$$4A - \begin{bmatrix} 1 & -2 \\ 4 & 5 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 30 \\ 29 & 63 \\ -19 & 15 \end{bmatrix}$$

Therefore

$$4A = \begin{bmatrix} 7 & 30 \\ 29 & 63 \\ -19 & 15 \end{bmatrix} + \begin{bmatrix} 1 & -2 \\ 4 & 5 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 28 \\ 33 & 68 \\ -22 & 16 \end{bmatrix}$$

$$A = \frac{1}{4}\begin{bmatrix} 8 & 28 \\ 33 & 68 \\ -22 & 16 \end{bmatrix} = \begin{bmatrix} 2 & 7 \\ 33/4 & 17 \\ -11/2 & 4 \end{bmatrix}.$$

QUESTION 4.

a) Express $z_1 = e^{-i\pi/2}$ and $z_2 = e^{-i\pi/6}$ in the form $z = a + ib$.

$$z_1 = \cos\frac{-\pi}{2} + i\sin\frac{-\pi}{2} = 0 + i(-1) = -i$$

$$z_2 = \cos\frac{-\pi}{6} + i\sin\frac{-\pi}{6} = \frac{\sqrt{3}}{2} - \frac{i}{2}$$

b) Express z_1/z_2 in the form $z = a + ib$.

$$\frac{z_1}{z_2} = \frac{e^{-i\pi/2}}{e^{-i\pi/6}} = e^{i(\pi/6-\pi/2)} = e^{-i\pi/3} = \frac{1}{2} - \frac{\sqrt{3}}{2}i$$

c) Express $\omega = -9 + 40i$ in the form $\omega = re^{i\theta}$.

We have $r = \sqrt{(-9)^2 + 40^2} = 41$. Since the vector is in the second quadrant, we have

$$\theta = \pi - \arctan\frac{40}{9} = 1.79211.$$

Thus,

$$\omega = 41e^{1.79211i}$$

d) Find $\omega\bar{\omega}$.

[1] We have $\bar{\omega} = 41e^{-1.79211i}$, so

$$\omega\bar{\omega} = 41e^{1.79211i}41e^{-1.79211i} = 1681.$$

Alternatively, we could just have multiplied out:

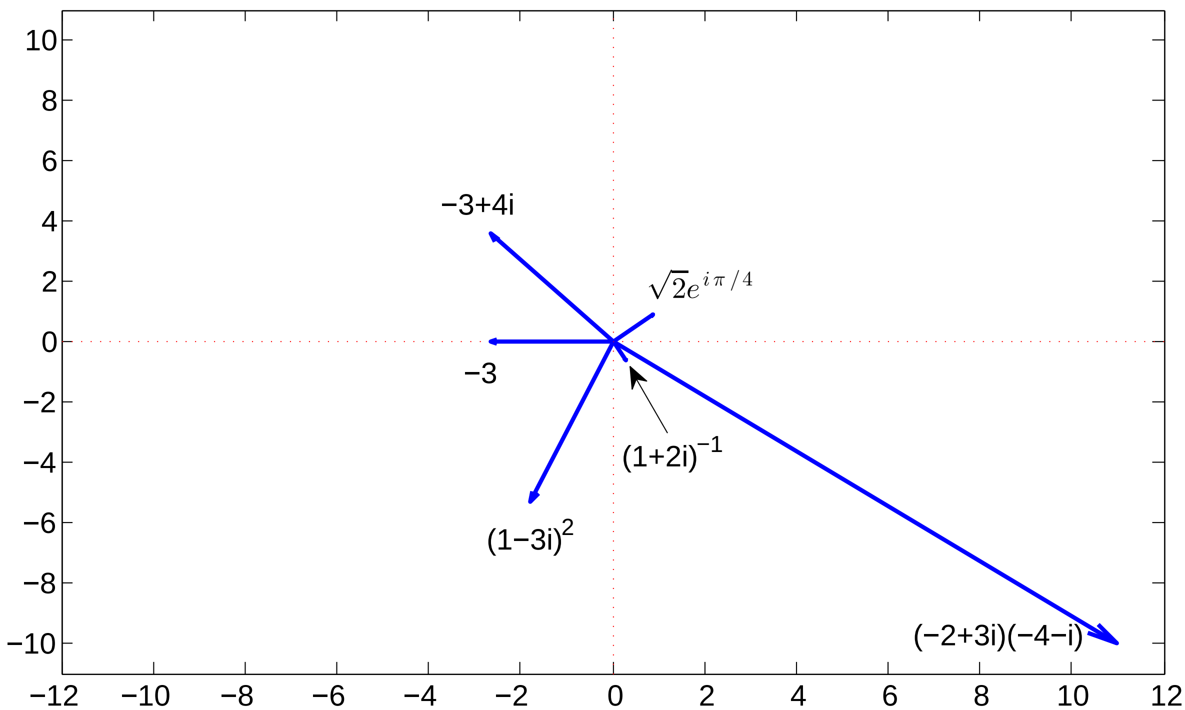
$$\omega\bar{\omega} = (-9 + 40i)(-9 - 40i) = 81 - 1600i^2 = 1681.$$

(1 mark for the answer. No part marks.)

QUESTION 5. Draw the following complex numbers in the plane:

$$-3+4i, \quad -3, \quad \sqrt{2}e^{\pi i/4}, \quad (-2+3i)(-4-i), \quad (1-3i)^2, \quad (1+2i)^{-1}.$$

[7] Solution: $\sqrt{2}e^{i\pi/4} = 1+i, \quad (-2+3i)(-4-i) = 11-10i, \quad (1-3i)^2 = -2-6i, \quad (1+2i)^{-1} = \frac{1}{3} - \frac{2}{3}i$



3 points for sketch, 1 point for finding the correct cartesian coordinates of each of $\sqrt{2}e^{i\pi/4}, \quad (-2+3i)(-4-i), \quad (1-3i)^2, \quad (1+2i)^{-1}$**.**