MAT 1332, Winter 2017, Assignment 3
Due Wednesday February 8 in the math department dropboxes by 7:00pm.
Late assignments will not be accepted; nor will unstapled assignments.
Professors in the math department will not lend you a stapler; do not ask for one.
Please print double sided to save paper.

Instructor (circle one): Guy Beaulieu Robert Smith? Xiaoying Wang

DGD (circle one): 1 2 3 4

Name (Prime student) ______________________ Student Number ____________________
Student Name ____________________________ Student Number ____________________
Student Name ____________________________ Student Number ____________________

By signing below, we declare that this work is our own, that we have not copied from any other individual or other source and that all students contributed equally.

Signatures ____

QUESTION 1. A population of peacocks in the wild is described by

$$\frac{dP}{dt} = P^3 - 2P^2 - 35P$$

a) The biologically meaningful equilibrium points are (circle all that apply):

i) $P = 0$ ii) $P = -5$ iii) $P = 7$ iv) $P = 5$.

b) The equilibrium $P = 7$ is (circle the correct answer): STABLE / UNSTABLE.

c) Draw the phase-line diagram here:

(Do not include any biologically meaningless solutions.)

d) If there are initially seven peacocks, what will happen eventually?
ANSWER: Nothing. Since 7 is an equilibrium, there will always be 7 peacocks.

[2] (a) Factoring, we have

$$P^3 - 2P^2 - 25P = P(P^2 - 2P - 35) = P(P+5)(P-7) = F(P)$$

Hence the equilibria are $P_1 = 0; P_2 = -5$ and $P_3 = 7$. The equilibrium P_2 is not biologically meaningful, as you can't have negative peacocks. You can, of course, have no peacocks, so both P_1 and P_3 are biologically meaningful. Hence: i) and iii).

(1 mark for each correct answer and -1 for each incorrect answer.)

[1] (b) Differentiating, we have

$$F'(P) = 3P^2 - 4P - 35.$$

(1 mark for the correct answer. No part marks.)

Substituting the biologically meaningful equilibrium points, we have

$$F'(0) = -35 < 0 \text{ so this point is stable}$$
$$F'(7) = 84 > 0 \text{ so this point is unstable}$$

[2] (c) The phase-line digram should include 0 and all positive numbers, but have no negative solutions (i.e., no arrows below zero).

(0.5 for each equilibrium, 0.5 for a correct arrow between the two equilibria and 0.5 for a correct arrow above 7. Lose 1 mark if there are contradictory arrows. Subtract 2 marks for the Life Sciences penalty if arrows are drawn below zero.)

(d) Since 7 is an equilibrium, there will always be 7 peacocks.

QUESTION 2. A bacterial culture starts with a certain amount of bacteria and expands at a rate proportional to its amount. After 3 hours, the population has quadrupled.

a) Determine the expression for the number of bacteria as a function of time.

b) Is the ratio of bacteria after 24 hours to the amount of bacteria after 8 hours the same as the ratio of the bacteria after three hours to the initial amount? YES / NO

c) How long until the population quadruples again?

(a) Let B represent the bacteria. We have $\frac{dB}{dt} = KB$, so separating variables and integrating gives

$$\frac{dB}{B} = K dt$$
$$\ln B = Kt + C$$
$$B = Ae^{Kt}$$

We don't know the initial amount, so let's call it B_0

Applying the conditions $B(0) = B_0$ and $B(3) = 4B_0$ gives

$$4B_0 = B_0 e^{3K}$$
$$e^{3K} = 4$$
$$3K = \ln 4$$
$$K = \frac{1}{3}\ln 4 = 0.462$$

Hence the solution is $B(t) = B_0 e^{0.462t}$

(b) After 8 hours, there are $P(8) = B_0 e^{0.462\times 8} = 40.317B_0$ bacteria.

After 24 hours, there are $P(24) = B_0 e^{0.462\times 24} = 65381.852B_0$ bacteria. The ratio is $P(24)/P(8) = 1621.675$, while the other ratio is $P(3)/P(0) = 3$. So they are not the same.

(c) We have

$$16B_0 = B_0 e^{0.462\bar{t}}$$
$$\bar{t} = \frac{\ln 16}{0.462} = 6 \text{ hours}$$

QUESTION 3. Carbon-14 has a half-life of approximately 5730 years. The decay rate of carbon-14 is proportional to the quantity present. A fossil has 0.03% carbon-14 compared to an equivalent living sample. Could that fossil be a dinosaur? (Circle one)

A) It is definitely a dinosaur.

B) It might be a dinosaur.

C) It cannot be a dinosaur.

D) It's impossible to tell.

[2] Since the carbon decays, the differential equation is negative. Hence we have $\frac{dQ}{dt} = -kQ$, where $Q(t)$ is the quantity of carbon-14 remaining at time t.

Separating variables and integrating (just as in Question 1), we have

$$Q = Ae^{-kt}$$

The initial condition gives $Q(0) = Q_0$, while the halflife condition says that $Q(5730) = Q_0 e^{-5730k} = \frac{Q_0}{2}$, so it follows that

$$k = -\frac{1}{5730}\ln\frac{1}{2} = \frac{\ln 2}{5730}.$$

Hence the solution is

$$Q(t) = Q_0 \exp\left(-\frac{\ln 2}{5730}t\right)$$

Our fossil has 0.03% carbon-14 remaining, so we have

$$\begin{aligned} 0.0003Q_0 &= Q_0 \exp\left(-\frac{\ln 2}{5730}\bar{t}\right) \\ \bar{t} &= \frac{5730\ln 0.0003}{-\ln 2} \\ &= 67{,}056 \text{ years.} \end{aligned}$$

That is, this fossil is 65 thousand years old. Since the dinosaurs died out 65 million years ago, there is no way this fossil can be a dinosaur. The answer is C.

(2 marks for the correct answer. No part marks. No marks if more than one answer is circled.)

QUESTION 4. Solve the following differential equations:

a) $\dfrac{dy}{dx} = \dfrac{e^{7x} - 2}{e^{3x+4y}}$

[3] Separating variables and integrating, we have

$$\begin{aligned} e^{4y} dy &= \frac{e^{7x} - 2}{e^{3x}} dx \\ \int e^{4y} dy &= \int \frac{e^{7x} - 2}{e^{3x}} dx \\ &= \int e^{4x} - 2e^{-3x} dx \\ \frac{e^{4y}}{4} &= \frac{e^{4x}}{4} - \frac{2e^{-3x}}{-3} + c \\ e^{4y} &= e^{4x} + \frac{8}{3} e^{-3x} + K \\ y &= \frac{1}{4} \ln \left(e^{4x} + \frac{8}{3} e^{-3x} + K \right) \end{aligned}$$

(Note that the constant cannot be moved outside the logarithm.)

(0.5 marks for separation of variables, 0.5 marks for rearranging fraction, 1 mark for integration, 0.5 for including the constant, 0.5 for keeping the constant inside the logarithm)

b) $y^4 \dfrac{dy}{dx} = 6x^5 y^5 - 42x^5$

[2] Factoring the right-hand side, then separating variables and integrating, we have

$$\begin{aligned} y^4 \frac{dy}{dx} &= 6x^5 (y^5 - 7) \\ \frac{y^4}{y^5 - 7} dy &= 6x^5 dx \\ \frac{1}{5} \int \frac{5y^4}{y^5 - 7} dy &= \int 6x^5 dx \\ \frac{1}{5} \ln |y^5 - 7| &= x^6 + c \\ \ln |y^5 - 7| &= 5x^6 + K \\ y^5 - 7 &= Ae^{5x^6} \\ y^5 &= Ae^{5x^6} + 7 \\ y &= \sqrt[5]{Ae^{5x^6} + 7} \end{aligned}$$

(0.5 marks for integrating, 0.5 for absolute value signs in the logarithm, 0.5 for keeping the 7 inside the root, 0.5 for finding y)

c) $y' = \dfrac{\tan(\pi y)}{t}$. Find the particular solution satisfying $y(1) = -\dfrac{1}{6}$. (You must find y explicitly.)

First rewrite the differential equation as $\dfrac{dy}{dt} = \dfrac{1}{t}\dfrac{\sin(\pi y)}{\cos(\pi y)}$. Then, separating variables and integrating, we have

$$\begin{aligned} \frac{\cos(\pi y)}{\sin(\pi y)} dy &= \frac{1}{t} dt \\ \int \frac{\cos(\pi y)}{\sin(\pi y)} dy &= \int \frac{1}{t} dt \end{aligned}$$

Let $u = \sin(\pi y)$. Then $\dfrac{du}{dy} = \pi \cos(\pi y)$ so $dy = \dfrac{du}{\pi \cos(\pi y)}$. We thus have

$$\begin{aligned} \int \frac{\cos(\pi y)}{u} \cdot \frac{du}{\pi \cos(\pi y)} &= \int \frac{1}{t} dt \\ \frac{1}{\pi} \int \frac{du}{u} &= \int \frac{1}{t} dt \\ \frac{1}{\pi} \ln |u| &= \ln |t| + c \\ \frac{1}{\pi} \ln |\sin(\pi y)| &= \ln |t| + c. \end{aligned}$$

Applying the initial condition, we have

$$\begin{aligned} \frac{1}{\pi} \ln \left| \sin\left(-\frac{\pi}{6}\right) \right| &= \ln 1 + c \\ \frac{1}{\pi} \ln \left| -\frac{1}{2} \right| &= c \\ c &= \frac{1}{\pi} \ln \frac{1}{2} \\ &= -\frac{\ln 2}{\pi} \end{aligned}$$

Solving for y, we have

$$\begin{aligned} \frac{1}{\pi} \ln |\sin(\pi y)| &= \ln |t| - \frac{\ln 2}{\pi} \\ \ln |\sin(\pi y)| &= \pi \ln |t| - \ln 2 \\ \ln |\sin(\pi y)| &= \ln |t|^{\pi} - \ln 2 \\ \sin(\pi y) &= \pm \exp(\ln |t|^{\pi} - \ln 2) \\ &= \pm \exp(\ln |t|^{\pi}) \exp(-\ln 2) \\ &= \pm \frac{|t|^{\pi}}{2} \end{aligned}$$

Only the negative root will give the correct initial conditon, so the solution is

$$y = \frac{1}{\pi} \arcsin\left(-\frac{|t|^{\pi}}{2}\right).$$