MAT 1332, Winter 2017, Assignment 2

Due Wednesday February 1 in the math department dropboxes by 7:00pm.

Late assignments will not be accepted; nor will unstapled assignments.

Professors in the math department will not lend you a stapler; do not ask for one.

Please print double sided to save paper.

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Name (Prime student) ______________________ Student Number ____________________

Student Name ____________________________ Student Number ____________________

Student Name ____________________________ Student Number ____________________

By signing below, we declare that this work is our own, that we have not copied from any other individual or other source and that all students contributed equally.

Signatures ____

Question 1. For each of the following integrals do the following:
(i) Explain in one sentence why the integral is an improper integral.
(ii) Decide whether the integral converges or diverges. If it converges, state what it converges to. Justify your answer.

$$\text{(a)} \int_0^3 \frac{e^{1/t^2}}{t^3} dt \qquad \text{(b)} \int_{-2}^2 \frac{8x^3+36x^2+54x+27}{(2x+3)^2} dx \qquad \text{(c)} \int_2^7 \ln(7-x)dx\,.$$

(i) (a)
$\displaystyle\int_0^3 \frac{e^{1/t^2}}{t^3} dt$ is an improper integral since $f(t) = \dfrac{e^{1/t^2}}{t^3}$ is continuous on $(0,3]$ but not continuous on $[0,3]$.

(i) (b)

[1] $\displaystyle\int_{-2}^2 \frac{8x^3+36x^2+54x+27}{(2x+3)^2} dx$ is an improper integral since $g(x) = \dfrac{8x^3+36x^2+54x+27}{(2x+3)^2}$ has a vertical asymptote at $x = \dfrac{3}{2}$, which is contained in the interval $[-2,2]$. (Note that even though the function ultimately simplifies, it's still an improper integral.)

(1) (c)
$\displaystyle\int_2^7 \ln(7-x)dx$ is an improper integral since $h(x) = \ln(7-x)$ is continuous on $[2,7)$ but not continuous on $[2,7]$.

(ii) (a)

[2] Let $u = \frac{1}{t^2}$, so $\frac{du}{dt} = -\frac{1}{t^3}$ and hence $dt = \frac{1}{2}t^3 du$. Then we have

$$\begin{aligned}
\int_0^3 \frac{e^{1/t^2}}{t^3} dx &= \int_{t=0}^{t=3} \frac{e^u}{t^3} \cdot -\frac{t^3}{2} du \\
&= -\frac{1}{2} \int_{t=0}^{t=3} e^u du \\
&= -\frac{1}{2} e^u \Big|_{t=0}^{t=3} \\
&= -\frac{1}{2} e^{1/t^2} \Big|_0^3 \\
&= -\frac{1}{2} e^{-1/9} + \frac{1}{2} \lim_{\epsilon \to 0^+} e^{1/\epsilon} \\
&= -\frac{1}{2} e^{-1/9} + \infty
\end{aligned}$$

It follows that this integral DIVERGES.

(1 mark for integration, 1 for conclusion in words)

(ii) (b)

This clearly isn't an integration by parts question (as it's not a product), and it's not really a substitution question either (as the obvious substitutions have derivatives that don't cancel). It's not a partial fractions question either, since the denominator is already in its basic form. This means the only remaining choice is to manipulate the function.

Either by factoring or long division, we can rewrite the integral as

$$\begin{aligned}
\int_{-2}^2 \frac{8x^3 + 36x^2 + 54x + 27}{(2x+3)^2} dx &= \int_{-2}^2 \frac{(2x+3)^3}{(2x+3)^2} dx \\
&= \int_{-2}^2 (2x+3) dx \\
&= \Big[x^2 + 3x\Big]_{-2}^2 \\
&= (4+6) - (4-6) \\
&= 12
\end{aligned}$$

(Note that the discontinuity becomes irrelevant after simplifying, so we don't need to split the integral limits at $\frac{3}{2}$, although it's fine if you do of course.) It follows that this integral CONVERGES to 12.

(ii) (c)

[3] Using integration by parts, we have

$$\begin{aligned}
u &= \ln(7-x) & v' &= 1 \\
u' &= -\frac{1}{7-x} & v &= x
\end{aligned}$$

Then the integral is

$$\begin{aligned}
\lim_{T\to 7^-}\left[x\ln(7-x)\Big|_2^T+\int_2^T\frac{x}{7-x}dx\right] &= \lim_{T\to 7^-}\left[x\ln(7-x)\Big|_2^T+\int_2^T\frac{x-7+7}{7-x}dx\right]\\
&= \lim_{T\to 7^-}\left[x\ln(7-x)\Big|_2^T+\int_2^T\left(-1+\frac{7}{7-x}\right)dx\right]\\
&= \lim_{T\to 7^-}\left[x\ln(7-x)-x-7\ln(7-x)\right]_2^T\\
&= \lim_{T\to 7^-}[(T-7)\ln(7-T)-T]-[-5\ln 5-5]\\
&= \lim_{T\to 7^-}\left[\frac{\ln(7-T)}{\frac{1}{T-7}}-T\right]-[-5\ln 5-5]\\
&\overset{L'H}{=} \lim_{T\to 7^-}\left[\frac{\frac{-1}{7-T}}{\left(\frac{1}{T-7}\right)^2}-T\right]-[-5\ln 5-5]\\
&= \lim_{T\to 7^-}[(T-7)-T]-[-5\ln 5-5]\\
&= -7+5\ln 5+2\\
&= 5\ln 5-5.
\end{aligned}$$

Hence the integral converges to $5\ln 5-5$.

(1 mark for integration, 1 for L'Hopital's rule, 1 for conclusion and value.)

QUESTION 2. For the following indefinite integrals

(i) $\displaystyle\int\frac{2x^4-26x^3+72x^2-4x+1}{x^2-13x+36}dx$ (ii) $\displaystyle\int\frac{x^3-16x^2+68x-27}{x^2-16x+64}dx$ (iii) $\displaystyle\int\frac{2x^2-x-6}{(x^2+4)(x-1)}dx$,

which of these terms is **NOT** present in the answer?

(i) A. $\frac{2}{3}x^3$ B. $-7\ln(x-9)$ C. $+C$ D. All are present. ANSWER: *B*

(ii) A. $\frac{1}{2}x^2$ B. $-\frac{5}{x-8}$ C. $5\ln(x-8)^2$ D. All are present. ANSWER: *C*

(iii) A. $\frac{3}{2}\ln(x^2+4)$ B. $\arctan x$ C. $\arctan\left(\frac{x}{2}\right)$ D. All are present. ANSWER: *B*

(i)

(2 marks for each answer. No part marks.)

[2] Since the degree of the numerator is larger than the degree of the denominator, we need to do long division:

$$\begin{array}{r|l}
 & 2x^2 \\
\hline
x^2-13x+36 & 2x^4-26x^3+72x^2-4x+1 \\
 & 2x^4-26x^3+72x^2 \\
\hline
 & -4x+1
\end{array}$$

Hence $2x^4 - 26x^3 + 72x^2 - 4x + 1 = 2x^2(x^2 - 13x + 36) - 4x + 1$ and so

$$\begin{aligned}\frac{2x^4 - 26x^3 + 72x^2 - 4x + 1}{x^2 - 13x + 36} &= \frac{2x^2(x^2 - 13x + 36) - 4x + 1}{x^2 - 13x + 36} \\ &= 2x^2 + \frac{1 - 4x}{x^2 - 13x + 36} \\ &= 2x^2 + \frac{1 - 4x}{(x-9)(x-4)}\end{aligned}$$

Using partial fractions, we have

$$\begin{aligned}\frac{1 - 4x}{(x-9)(x-4)} &= \frac{A}{x-9} + \frac{B}{x-4} \\ 1 - 4x &= A(x-4) + B(x-9) \\ x = 4: \qquad -15 &= B(-5) \qquad B = 3 \\ x = 9: \qquad -35 &= A(5) \qquad A = -7\end{aligned}$$

Hence the integral is

$$\int \left(2x^2 - \frac{7}{x-9} + \frac{3}{x-4}\right) dx = \frac{2}{3}x^3 - 7\ln|x-9| + 3\ln|x-4| + c$$

The term that isn't present is $-7\ln(x-9)$, because the absolute value signs are crucial. The answer is B.

(ii)

[2] Once again, we need to do long division:

$$\begin{array}{r|l} & x \\ \hline x^2 - 16x + 64 & x^3 - 16x^2 + 68x - 27 \\ & x^3 - 16x^2 + 64x \\ \hline & \qquad\qquad 4x - 27 \end{array}$$

Hence $x^3 - 16x^2 + 68x - 27 = x(x^2 - 16x + 64) + 4x - 27$, so we have

$$\frac{x^3 - 16x^2 + 68x - 27}{x^2 - 16x + 64} = x + \frac{4x - 27}{x^2 - 16x + 64}$$

Note that the denominator factors as $x^2 - 16x + 64 = (x-8)^2$. Hence, using partial fractions, we have

$$\begin{aligned}\frac{4x - 27}{x^2 - 16x + 64} &= \frac{A}{x-8} + \frac{B}{(x-8)^2} \\ 4x - 27 &= A(x-8) + B \\ x = 8: \qquad 5 &= B \\ x = 0: \qquad -27 &= A(-8) + 5 \\ 27 &= 8A - 5 \qquad A = 4\end{aligned}$$

Thus

$$\int \frac{x^3 - 16x^2 + 68x - 27}{x^2 - 16x + 64} dx = \int x dx + \int \frac{4}{x-8} + \int \frac{5}{(x-8)^2} dx$$
$$= \frac{1}{2}x^2 + 4\ln|x-8| - \frac{5}{x-8} + C$$

The term that isn't present is $5\ln(x-8)^2$, because $\int \frac{1}{f(x)} dx \neq \ln f(x)$ unless $f(x) = x$. The answer is C.

(iii)

[2] Since the degree of the numerator is strictly less than the degree of the denominator, we don't need long divison in this case. Note that the denominator is already factored as much as it can be.

Using partial fractions, we have

$$\frac{2x^2 - x - 6}{(x^2+4)(x-1)} = \frac{Ax+B}{x^2+4} + \frac{C}{x-1}$$
$$2x^2 - x - 6 = (Ax+B)(x-1) + C(x^2+4)$$
$$x = 1: \quad -5 = C(5) \qquad C = -1$$
$$x = 0: \quad -6 = B(-1) - 4$$
$$6 = B + 4 \qquad B = 2$$
$$x = 2: \quad 5 - 2 - 6 = (2A+B)(1) + C(8)$$
$$0 = 2A + 2 - 8$$
$$2A = 6$$
$$A = 3$$

The integral is thus

$$\int \left(\frac{3x+2}{x^2+4} - \frac{1}{x-1}\right) dx = \int \left(\frac{3x}{x^2+4} + \frac{2}{x^2+4} - \frac{1}{x-1}\right) dx$$
$$= \int \frac{3x}{x^2+4} dx + \frac{1}{2}\int \frac{1}{\left(\frac{x}{2}\right)^2 + 1} dx - \int \frac{1}{x-1} dx$$

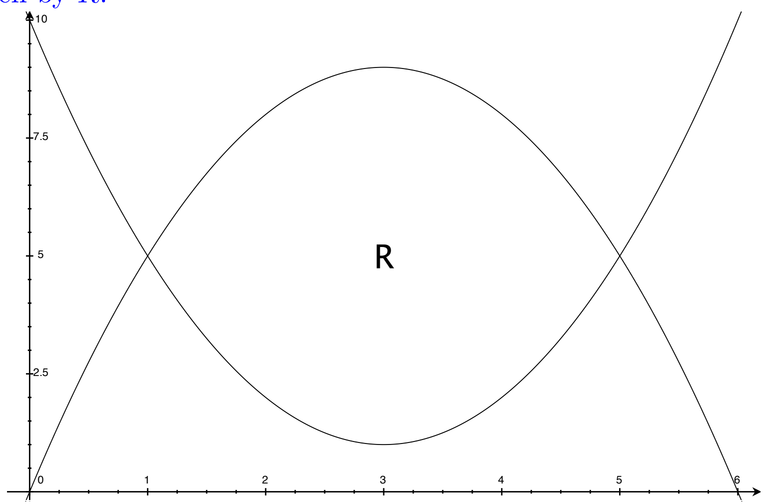
where we divided through by 4 in the second integral. For the first integral, we'll use the substitution $u = x^2 + 4$ so that $\frac{du}{dx} = 2x$ or $dx = \frac{du}{2x}$. For the second integral, we'll use the substitution $w = \frac{x}{2}$ so that $\frac{dw}{dx} = \frac{1}{2}$ or $dx = 2dw$. We thus have

$$\int \frac{3x}{u}\frac{du}{2x} + \frac{1}{2}\int \frac{1}{w^2+1} 2dw - \int \frac{1}{x-1} dx = \frac{3}{2}\int \frac{1}{u} du + \int \frac{1}{w^2+1} dw - \int \frac{1}{x-1} dx$$
$$= \frac{3}{2}\ln|u| + \arctan w - \ln|x-1| + C$$
$$= \frac{3}{2}\ln|x^2+4| + \arctan\left(\frac{x}{2}\right) - \ln|x-1| + C.$$

Note that, since $x^2 > 0$, the expression $x^2 + 4$ is always positive, so $\ln|x^2+4| = \ln(x^2+4)$. The absolute value signs are not needed here (although they are for the other logarithm). It follows that the term $\arctan(x)$ is not present in the answer, as you have to resubstitute. The answer is B.

Question 3. Sketch the region bounded by $y = x^2 - 6x + 10$ and $y = -x^2 + 6x$. Find the volume of the object created when this region is rotated around the x-axis. Leave your answer in exact form (not a decimal approximation).

There is only one region bounded by the two curves (all other possibilities are unbounded), given by R:



We need to find the intercepts to determine the limits of integration. This leads to:

$$\begin{aligned} x^2 - 6x + 10 &= -x^2 + 6x \\ 2x^2 - 12x + 10 &= 0 \\ x^2 - 6x + 5 &= 0 \\ (x-5)(x-1) &= 0 \end{aligned}$$

and hence the intercepts are 1 and 5. The downward-facing quadratic is larger than the upward-facing quadratic in this range, so we have

$$\begin{aligned} V &= \int_1^5 \pi[f_2(x)^2 - f_2(x)^2]dx \\ &= \int_1^5 \pi[(-x^2+6x)^2 - (x^2-6x+10)^2]dx \\ &= \pi \int_1^5 \left[(x^4 - 12x^3 + 36x^2) - (x^4 + 36x^2 + 100 - 12x^3 + 20x^2 - 120x)\right] dx \\ &= \pi \int_1^5 \left[-100 - 20x^2 + 120x)\right] dx \\ &= \pi \left[-100x - \frac{20x^3}{3} + 60x^2\right]_1^5 \\ &= \pi \left[-500 - \frac{2500}{3} + 1500 - \left(-100 - \frac{20}{3} + 60\right)\right] \\ &= \frac{640}{3}\pi \text{ units}^2 \end{aligned}$$

QUESTION 4. A patch of moss initially occupies an area of 20 square centimetres. The moss grows at a rate inversely proportional to the size of the patch. After 250 hours, the patch has grown to 30 square centimetres.

(a) Find the size of the patch as a function of time.

(b) How large is the patch after 300 hours? (to one decimal place)

(c) How long does it take until the moss reaches 50 square centimetres?

(a) TRUE or FALSE, the solution is $P(t) = \pm\sqrt{2t+400}$. FALSE.
If false, write the correct solution here: $P(t) = \sqrt{2t+400}$
(1 mark for FALSE, 1 for the correct solution)

[2] The differential equation that describes this phenomenon is

$$\frac{dP}{dt} = \frac{k}{P}$$

Separating variables, we have

$$\begin{aligned} PdB &= kdt \\ \int PdP &= \int kdt \\ \frac{1}{2}P^2 &= kt + c \\ P &= \sqrt{2kt + 2c} \end{aligned}$$

(choosing the positive root since moss can't have a negative area). Applying the initial condition, we have

$$\begin{aligned} 20 &= \sqrt{2k(0) + 2c} \\ 400 &= 2c \\ c &= 200 \end{aligned}$$

After 250 hours, we have

$$\begin{aligned} 30 &= \sqrt{2k(250) + 400} \\ 900 &= 500k + 400 \\ k &= 1 \\ P(t) &= \sqrt{2t + 400} \end{aligned}$$

The answer is FALSE because the negative solution is not included.

(b) TRUE or FALSE, the answer is 31.6. FALSE.
If false, write the correct solution here: 31.6 cm^2.
(1 mark for FALSE, 1 mark for the units)
[2] After 300 hours, we have

$$P(300) = \sqrt{2(300) + 400} = \sqrt{1000} = 31.6 \text{ cm}^2$$

The solution is incorrect without units.

(c) TRUE or FALSE, the answer is 1050 hours. TRUE
If false, write the correct solution here:
(1 mark for TRUE. No part marks.)
[1] We have

$$\begin{aligned} 50 &= \sqrt{2\bar{t} + 400} \\ 2500 &= 2\bar{t} + 400 \\ \bar{t} &= 1050 \text{ hours} \end{aligned}$$

(Note that the units are part of the solution.)