

MAT 1332, Winter 2017, Assignment 1

Due Wednesday January 25 in the math department dropboxes by 7:00pm.

Late assignments will not be accepted; nor will unstapled assignments.

Professors in the math department will not lend you a stapler; do not ask for one.

Please print double sided to save paper.

Instructor (circle one): Guy Beaulieu Robert Smith? Xiaoying Wang
 DGD (circle one): 1 2 3 4
 Name (Prime student) _____ Student Number _____
 Student Name _____ Student Number _____
 Student Name _____ Student Number _____

By signing below, we declare that this work is our own, that we have not copied from any other individual or other source and that all students contributed equally.

Signatures _____

QUESTION 1. Calculate each of the following definite integrals.

(a) $\int_1^3 \frac{(\ln(x))^2}{2x} dx$

Let $u = \ln(x)$. Then $du = \frac{1}{x} dx$, $dx = x du$. Now substitute for lower/upper limits. Then $x = 1 \Rightarrow u = \ln(1) = 0$ and $x = 3 \Rightarrow u = \ln(3)$. Then

$$\begin{aligned} \int_1^3 \frac{(\ln(x))^2}{2x} dx &= \frac{1}{2} \int_0^{\ln(3)} u^2 du \\ &= \frac{1}{2} \frac{u^3}{3} \Big|_0^{\ln(3)} \\ &= \frac{1}{6} (\ln(3))^3. \end{aligned}$$

(b) $\int_0^{\frac{\pi}{2}} \cos(x)e^{\sin(x)} dx$

Let $u = \sin(x)$. Then $du = \cos(x) dx$. Changing limits gives $x = 0 \Rightarrow u = 0$, $x = \frac{\pi}{2} \Rightarrow u = 1$. Then

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \cos(x)e^{\sin(x)} dx &= \int_0^1 e^u du \\ &= e^u \Big|_0^1 \\ &= e - 1. \end{aligned}$$

(c) $\int_0^{\pi/4} \frac{\sin(x)}{\cos^3(x)} dx$

Let $u = \cos(x)$. Then

$$\begin{aligned} \int_0^{\pi/4} \frac{\sin(x)}{\cos^3(x)} dx &= \int_{x=0}^{x=\pi/4} -\frac{1}{u^3} du \\ &= \frac{1}{2} u^{-2} \Big|_{x=0}^{x=\pi/4} \\ &= \frac{1}{\cos^2(x)} \Big|_0^{\pi/4} \\ &= \frac{1}{2} (2 - 1) \\ &= \frac{1}{2}. \end{aligned}$$

QUESTION 2. Calculate each of the following definite integrals.

(a) $\int_1^3 \frac{\ln(x)}{x^2} dx$

Let $u = \ln(x)$, $v' = x^{-2}$. Then $u' = \frac{1}{x}$ and $v = -x^{-1}$. Then, by integration by parts, we have

$$\begin{aligned} \int_1^3 \frac{\ln(x)}{x^2} dx &= \ln(x) \left(-\frac{1}{x} \right) \Big|_1^3 - \int_1^3 \frac{1}{x} \left(-\frac{1}{x} \right) dx \\ &= -\frac{1}{x} \ln(x) \Big|_1^3 + \int_1^3 x^{-2} dx \\ &= -\left(\frac{\ln(3)}{3} - \ln(1) \right) + \frac{x^{-1}}{-1} \Big|_1^3 \\ &= \frac{2}{3} - \frac{\ln(3)}{3}. \end{aligned}$$

(b) $\int_1^2 x^2 e^{-x} dx$

[3]

Let $u = x^2$, $v' = e^{-x}$. Then $u' = 2x$, $v = -e^{-x}$. Integration by parts:

$$\begin{aligned} \int_1^2 x^2 e^{-x} dx &= -x^2 e^{-x} \Big|_1^2 + \int_1^2 2x e^{-x} dx \\ &= -(4e^{-2} - e^{-1}) + 2 \int_1^2 x e^{-x} dx. \end{aligned}$$

We calculate $\int_1^2 x e^{-x} dx$ again by integration by parts. Let $u = x, v' = e^{-x}$. Then $u' = 1, v = -e^{-x}$. We thus have

$$\begin{aligned} \int_1^2 x e^{-x} dx &= -x e^{-x} \Big|_1^2 + \int_1^2 e^{-x} dx \\ &= -2e^{-2} + e^{-1} - e^{-x} \Big|_1^2 \\ &= -2e^{-2} + e^{-1} - e^{-2} + e^{-1}. \end{aligned}$$

Therefore

$$\begin{aligned} \int_1^2 x^2 e^{-x} dx &= -4e^{-2} + e^{-1} + 2(-2e^{-2} + e^{-1} - e^{-2} + e^{-1}) \\ &= 5e^{-1} - 10e^{-2}. \end{aligned}$$

(1 mark for the first integral, 1 mark for the second, 1 mark for the final answer.)

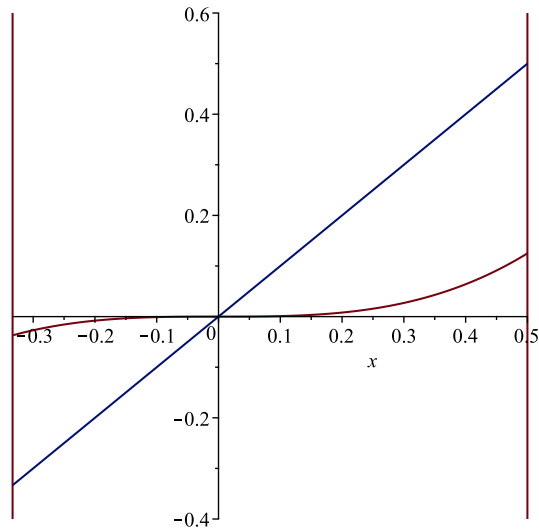
(c) $\int_{-\pi}^{\pi} [x^3 + 3x \sin(3x)] dx$

$$\begin{aligned} \int_{-\pi}^{\pi} [x^3 + 3x \sin(3x)] dx &= \int_{-\pi}^{\pi} x^3 dx + 3 \int_{-\pi}^{\pi} x \sin(3x) dx \\ &= \frac{x^4}{4} \Big|_{-\pi}^{\pi} + 3 \left(-\frac{x \cos(3x)}{3} \Big|_{-\pi}^{\pi} + \int_{-\pi}^{\pi} \frac{\cos(3x)}{3} dx \right) \\ &= 0 - (\pi \cos(\pi) + \pi \cos(\pi)) + \frac{1}{3} (\sin(3\pi) - \sin(-3\pi)) \\ &= 2\pi. \end{aligned}$$

QUESTION 3. Find and sketch the area enclosed by functions $y_1(x) = x^3$, $y_2(x) = x$, $x = -\frac{1}{3}$, and $x = \frac{1}{2}$.

We see that the area enclosed by functions consists of two parts. Let S be the total area and S_1, S_2 be the area in the third quadrant and first quadrant respectively. Then

$$\begin{aligned} S &= \int_{-\frac{1}{3}}^0 (x^3 - x) dx + \int_0^{\frac{1}{2}} (x - x^3) dx \\ &= \frac{x^4}{4} \Big|_{-\frac{1}{3}}^0 - \frac{x^2}{2} \Big|_{-\frac{1}{3}}^0 + \frac{x^2}{2} \Big|_0^{\frac{1}{2}} - \frac{x^4}{4} \Big|_0^{\frac{1}{2}} \\ &= 0 - \frac{1}{4} \left(\frac{1}{3}\right)^4 - \left(0 - \frac{1}{2} \left(\frac{1}{3}\right)^2\right) + \frac{1}{2} \left(\frac{1}{2}\right)^2 - \frac{1}{4} \left(\frac{1}{2}\right)^4 \\ &= \frac{839}{5184} = 0.1618. \end{aligned}$$



QUESTION 4. Find and sketch the area enclosed by functions $y_1(x) = -x^2 + 2x + 1$ and $y_2(x) = |x - 1|$.

Solving for $y_1(x) = y_2(x)$, we obtain intersections: $x = 0$ and $x = 2$. Let S be the total area, S_1 be the area for $0 \leq x \leq 1$ and S_2 be the area for $1 \leq x \leq 2$.

$$y_2(x) = \begin{cases} x - 1, & \text{if } x \geq 1, \\ 1 - x, & \text{if } x < 1. \end{cases}$$

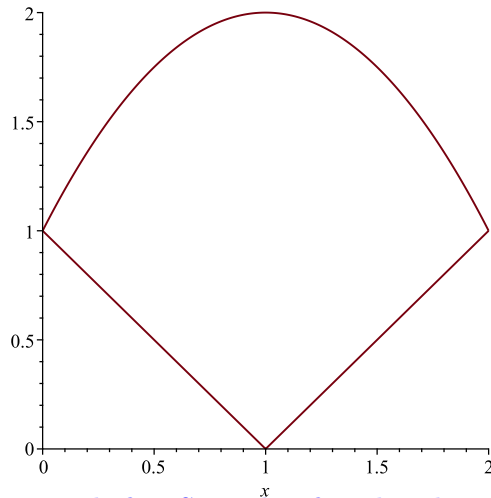
Then the area is given by

$$\begin{aligned} S_1 &= \int_0^1 (-x^2 + 2x + 1 - (1 - x)) \, dx \\ &= \int_0^1 (-x^2 + 2x + 1 - 1 + x) \, dx \\ &= \int_0^1 (-x^2 + 3x) \, dx \\ &= -\frac{x^3}{3} \Big|_0^1 + 3\frac{x^2}{2} \Big|_0^1 \\ &= \frac{7}{6}. \end{aligned}$$

Similarly,

$$S_2 = \int_1^2 (y_1(x) - y_2(x)) \, dx = \int_1^2 (-x^2 + x + 2) \, dx = \frac{7}{6}.$$

$S = S_1 + S_2 = \frac{7}{3}$. Or by symmetry, $S_1 = S_2$, $S = S_1 + S_2 = 2S_1 = \frac{7}{3}$.



(1 mark for finding S_1 , 1 mark for S_2 and 1 for the sketch. (If the argument is a symmetry one, that's acceptable if done in words.)

QUESTION 5. Assume that the growth of a tree obeys the rule $\frac{dx}{dt} = t - e^{-2t}$, where t is time in years and x measures the height of the tree above ground in metres. How much does the tree grow between Year 2 and Year 5?

$$\begin{aligned} \int_2^5 dx &= \int_2^5 (t - e^{-2t}) dt = \int_2^5 t dt - \int_2^5 e^{-2t} dt \\ &= \left. \frac{t^2}{2} \right|_2^5 + \frac{1}{2} \int_{-4}^{-10} e^u du \\ &= \frac{1}{2}(25 - 4) + \frac{1}{2}(e^{-10} - e^{-4}) \\ &= \frac{21}{2} + \frac{1}{2} \left(\frac{1}{e^{10}} - \frac{1}{e^4} \right) = 10.491. \end{aligned}$$

Therefore, the tree grows 10.491 metres between Year 2 and Year 5.

[1] QUESTION 6. We use Riemann sum with four rectangles of equal width to approximate $\int_0^1 (1 - x^2) dx$. Among the following statements, which one is correct? **Answer: B**

(A) The left Riemann sum of $\int_0^1 (1 - x^2) dx$ is $\frac{1}{4} \sum_{k=1}^4 \left(1 - \left(\frac{k}{4}\right)^2\right)$.

(B) The left Riemann sum of $\int_0^1 (1 - x^2) dx$ is $\frac{1}{4} \sum_{k=0}^3 \left(1 - \left(\frac{k}{4}\right)^2\right)$.

(C) The left Riemann sum of $\int_0^1 (1 - x^2) dx$ is $\frac{1}{4} \sum_{k=0}^4 \left(1 - \left(\frac{k}{4}\right)^2\right)$.

(D) The right Riemann sum of $\int_0^1 (1 - x^2) dx$ is $\frac{1}{4} \sum_{k=0}^3 \left(1 - \left(\frac{k}{4}\right)^2\right)$.