AM 2811 Midterm Solns

February 27, 2015 7:15 PM

7:15

Q1

(a) Use the isomorphism $ax^2+bx+c \leftrightarrow \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

Then $1-x-2x^2 \leftrightarrow \overset{\vec{v}_1}{\begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}}$, $5-4x-7x^2 \leftrightarrow \overset{\vec{v}_2}{\begin{pmatrix} -7 \\ -4 \\ 5 \end{pmatrix}}$, $-3+x \leftrightarrow \overset{\vec{v}_3}{\begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix}}$

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$$

$$\left[\begin{array}{ccc|c} -2 & -7 & 0 & 0 \\ -1 & -4 & 1 & 0 \\ 1 & 5 & -3 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 5 & -3 & 0 \\ -1 & -4 & 1 & 0 \\ -2 & -7 & 0 & 0 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 5 & -3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 3 & -6 & 0 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 5 & -3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

5/5

So $c_3=0$, and a basis is

$\left\{ \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -7 \\ -4 \\ 5 \end{bmatrix} \right\}$ or equivalently

$\{-2x^2-x+1, -7x^2-4x+5\}$

(b) $c_1 \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -7 \\ -4 \\ 5 \end{bmatrix} = \begin{bmatrix} a \\ 3 \\ -4 \end{bmatrix}$

$$\left[\begin{array}{cc|c} -2 & -7 & a \\ -1 & -4 & 3 \\ 1 & 5 & -4 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 5 & -4 \\ -1 & -4 & 3 \\ -2 & -7 & a \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 5 & -4 \\ 0 & 1 & -1 \\ 0 & 3 & a-8 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 5 & -4 \\ 0 & 1 & -1 \\ 0 & 0 & a-5 \end{array}\right]$$

4/4.

$\therefore$ need $a=5$ for $-4+3x+ax^2 \in \operatorname{span} S$.

(c) $\begin{pmatrix} c & d & c \\ d & c & d \end{pmatrix} = c \overset{M_1}{\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}} + d \overset{M_2}{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}}$

$= \operatorname{span}\left\{ \underset{M_1}{\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}}, \underset{M_2}{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}} \right\}$

Also $c M_1 + d M_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

3/3

$\Rightarrow c=0, \quad d=0$

so $\{M_1, M_2\}$ is a basis for the set

7:27.

Q2

7:33

(a) $L(\vec{u}+\vec{v}) = L(\vec{u}) + L(\vec{v})$ for $\vec{u},\vec{v} \in V$

$L(k\vec{v}) = k\,L(\vec{v})$

2/2

$L(k\vec{v}) = k\,L(\vec{v})$ ✓ ~12

(b) We show $\ker(L) = \{\vec{v} \in V : L(\vec{v}) = \vec{0}\}$ is a subspace of V. ✓

Let $\vec{0} \in V$ be the zero vector of V.
Then $L(\vec{0}+\vec{0}) = L(\vec{0}) = 2L(\vec{0})$ so $L(\vec{0}) = \vec{0}$ ✓
and $\vec{0} \in \ker(L)$.

Let $\vec{u}, \vec{v} \in \ker(L)$. 5/5.
Then $L(\vec{u}) = \vec{0}$, $L(\vec{v}) = \vec{0}$ ✓
So $L(\vec{u}+\vec{v}) = L(\vec{u}) + L(\vec{v}) = \vec{0}+\vec{0} = \vec{0}$ ✓
So $\vec{u}+\vec{v} \in \ker(L)$.
Also $L(c\vec{u}) = c\,L(\vec{u}) = c\,\vec{0} = \vec{0}$ ✓
so $c\vec{u} \in \ker L$ and
$\ker(L)$ is a subspace of V.

(c) Here $L(u) = u_{xx} + u_{yy}$

We only need to show L is a linear operator then use the result of (b). ✓

$$L(u+v) = (u+v)_{xx} + (u+v)_{yy}$$ ✓
$$= u_{xx} + v_{xx} + u_{yy} + v_{yy}$$ ✓
$$= L(u) + L(v)$$

$$L(ku) = (ku)_{xx} + (ku)_{yy}$$ ✓
$$= k\,u_{xx} + k\,u_{yy}$$ ✓ 3/3
$$= k\,L(u)$$

So L is a linear operator and by (b) $\ker(L)$ is a vector space.

(d) This has form $L(u) = 1$ ✓
So if $u=0$, then $L(0) = 1 = 0$
So 0 not a soln of $L(u) = 0$ 2/2
and this is not a vector space. ✓

7:42

Q3 7:51

(a) $Q = \left\{\begin{pmatrix} a & b \\ b & a\end{pmatrix} : b = 2a\right\} = \left\{\begin{pmatrix} a & 2a \\ 2a & a\end{pmatrix}\ a \in \mathbb{R}\right\}$ 3/3

$= \ \{ (1 \ \ 2) \}$

(a) $Q = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} : b = 2a \right\} = \left\{ \begin{pmatrix} a & 2a \\ 2a & a \end{pmatrix} \; a \in \mathbb{R} \right\}$

$= \operatorname{Span}\left\{ \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \right\}$

i.e. this is a VS with basis $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$.

Alternatively show its a subspace of $M_{2\times 2}$. 2

Note $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in Q$ (set $a=0$).

Also show closure under + and scalar mult.

3/3

(b) $F = \{ f : \mathbb{R} \to \mathbb{R},$ with $f(x+3) = f(x) \}$

is a subset of the VS $\mathcal{F}(\mathbb{R}, \mathbb{R})$.

The zero f^n $z(x) = 0$ is in F since

$$z(x+3) = 0 = z(x)$$

Also if $f, g \in F$ then

$$(cf + dg)(x+3) = cf(x+3) + dg(x+3)$$
$$= cf(x) + dg(x)$$
$$= (cf + dg)(x)$$ 2

so $cf + dg \in F$ for all $c, d \in \mathbb{R}$.

So F is a subspace.

3/3

(c) $G = \left\{ \vec{v} \in \mathbb{R}^2 : \vec{v} = \begin{pmatrix} x \\ 0 \end{pmatrix} \text{ or } \vec{v} = \begin{pmatrix} 0 \\ y \end{pmatrix} \right\}$

$x \in \mathbb{R}, \; y \in \mathbb{R}$

Note $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \in G$, $\begin{pmatrix} 0 \\ 1 \end{pmatrix} \in G$, but

$\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \notin G$.

So G is not a subspace of $\mathbb{R}^2$.

3/3

7:59

Q 4 8:03

(a) $\operatorname{rng}(B) = $ col space (B)

pivots

$$\begin{pmatrix} \boxed{1} & -3 & -7 & 9 \\ 0 & \boxed{1} & 5 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

So $\operatorname{rng}(B) = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ -2 \end{bmatrix} \right\}$

OR Use REF(B^T) rows give basis for rowsp(B^T)

$\to (1, 0, 1)$
$(0, 1, 1)$

Span$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

Cokernel $B = \ker B^T$

$= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$

4x3

$\frac{4}{4}$

$$x + z = 0$$
$$y + z = 0$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -z \\ -z \\ z \end{pmatrix} = z \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = \text{span}\left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \right\}$$

(b) $\dim \ker B + \text{rank } B = \#\text{ cols } B$

$= 4.$ $\frac{2}{2}$

(c) First reduce C to REF.

Then rank C = # pivots in REF $C = r$.

We can solve for r of the pivot variables leaving the remaining variables as parametric

There are $n-r$ remaining free variables. $\frac{4}{4}$

So $n-r + r = n$

or $\dim \ker C + \text{rank } C = n.$

8:16.

Q 5

5:39 (a) $\|f\| = \sqrt{\langle f, f\rangle} = \sqrt{\int_{-1}^{+1} f(x) f(x)\, e^x\, dx}$ $3/3.$

(b) $|\langle f, g\rangle| \leq \|f\|\,\|g\|$ $\frac{4}{4}$

$$\left|\int_{-1}^{+1} f(x)\, g(x)\, e^x\, dx\right| \leq \sqrt{\int_{-1}^{+1} f(x)^2 e^x\, dx}\,\sqrt{\int_{-1}^{+1} g(x)^2 e^x\, dx}$$

(c) $\left|\int_{-1}^{+1} x e^{-x} e^x\, dx\right| = \left|\int_{-1}^{+1} x\, dx\right| = \left|\left[\frac{x^2}{2}\right]_{-1}^{+1}\right| = 0.$

$$\int_{-1}^{+1} \overset{u}{x^2} \overbrace{e^x\, dx}^{dv} = [x^2 e^x]_{-1}^{+1} - \int_{-1}^{+1} \overset{u}{2x} \overbrace{e^x\, dx}^{dv}$$

$= e - e^{-1} - 2[[xe^x]_{-1}^{+1} - \int_{-1}^{+1} e^x dx]$ $\frac{3}{3}$

+1 Calc. $\int$'s

+1 $\int_{-1}^{+1} x e^{-x} e^x dx = 0$

$= e - e^{-1} - 2\{\cancel{e} + e^{-1} - (\cancel{e} - e^{-1})\}$

$= e - 5e^{-1}$

+2 other 2 $\int$'s

$= e - 5e^{-1}$

+2 other 2 $\int$'s.

$$\int_{-1}^{+1} e^{-2x} e^{x}\, dx = \int_{-1}^{+1} e^{-x}\, dx = \left[-e^{-x}\right]_{-1}^{+1} = e - e^{-1}$$

Cauchy Schwarz is:

$0 \leq \sqrt{e - 5e^{-1}}\,\sqrt{e - e^{-1}}$ which is correct

(d) Not necessarily true.

E.g. consider $\begin{pmatrix}1\\1\end{pmatrix}\cdot\begin{pmatrix}-1\\1\end{pmatrix} = 0$ in the dot product

But in a weighted inner product

$$\left\langle \begin{pmatrix}u_1\\u_2\end{pmatrix}, \begin{pmatrix}v_1\\v_2\end{pmatrix} \right\rangle = w_1 u_1 v_1 + w_2 u_2 v_2$$

$$\left\langle \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}-1\\1\end{pmatrix} \right\rangle = w_1(-1) + w_2(1) = 0$$

only if $w_2 - w_1 = 0$

3/3

Q6 6:00 pm

(a) $c_1 \sin(x) + c_2 \sin(2x) + c_3 \sin(3x) + \dots = f(x)$

$$\Rightarrow \langle \sin mx,\ c_1 \sin(x) + c_2 \sin(2x) + \dots \rangle = \langle \sin mx, f(x) \rangle$$

$$c_m \langle \sin(mx), \sin(mx) \rangle = \langle \sin(mx), f(x) \rangle$$

since $\langle \sin(mx), \sin(nx) \rangle = 0$ for $m \neq n$

$$\Rightarrow c_m = \frac{\langle \sin(mx), f(x) \rangle}{\langle \sin(mx), \sin(mx) \rangle} \quad (*)$$

4/4

(b) Suppose $c_1 \sin(x) + c_2 \sin(2x) + c_3 \sin(3x) + \dots = 0$

Then by $(*)$

$$c_m = \frac{\langle \sin(mx), 0 \rangle}{\langle \sin(mx), \sin(mx) \rangle} = 0$$

2/2

So $\sin(x), \sin(2x), \dots$ are linearly independent.

6:04 pm.