University of Western Ontario
London, Ontario
Applied Mathematics 2811b
Midterm Examination - Friday February 27 , 2015

Time: 7:00 pm – 9:30 pm **P&AB 106 and 148**

Name:

Instructions:

- This exam contains two pages. No calculators allowed.
- Write your name, section number and answers in the exam booklets provided.
- To gain full credit you must completely justify your answers.
- An info sheet of one quarter of an 8.5×11 sheet (front side only) of handwritten notes is allowed. Handwriting and equations should be no smaller than that on page 297 of Olver and Shakiban; and have no more than 5 lines per inch. Failure to meet these criteria and to hand in your info sheet with your name on it will incur a penalty.
- Return this exam paper and your info sheet with your answer booklet.

1. Answer the following

   (a) Determine a basis for the set of polynomials $S = \{1 - x - 2x^2,\ 5 - 4x - 7x^2,\ -3 + x\}$, under the usual addition and scalar multiplication of polynomials in $\mathcal{P}^{(2)}(x)$.

   (b) For what value of a is $-4 + 3x + ax^2$ in the span of S?

   (c) You can assume that the set of matrices $\left\{ \begin{pmatrix} c & d & c \\ d & c & d \end{pmatrix} : c, d \in \mathbb{R} \right\}$ is a vector space. Give a basis for this vector space.

2. Suppose that $L : V \longrightarrow W$ is a linear operator and V and W are vector spaces, with $\mathbf{f} \in W$.

   (a) What properties does L have to satisfy?

   (b) Show that $\{\mathbf{v} \in V : L(\mathbf{v}) = \mathbf{0}\}$ is a vector space.

   (c) Show that $\{u \in C^2(\mathbb{R}^2, \mathbb{R}) : (\frac{\partial}{\partial x})^2 u + (\frac{\partial}{\partial y})^2 u = 0\}$ is a vector space (i.e. show that the solutions of $u_{xx} + u_{yy} = 0$ form a vector space). You can assume that $C^2(\mathbb{R}^2, \mathbb{R})$ is a vector space.

   (d) Why do the solutions of $u_{xx} + u_{yy} = 1$ not form a vector space?

Over $\Longrightarrow$

3. You can assume that $\mathbb{R}^n$, $\mathcal{M}_{m\times n}$ and $\mathcal{F}(S,V)$ (the functions from S to a vector space V) are vector spaces. Determine whether the following are or are not vector spaces. As usual completely justify your answers.

   (a) The set of matrices $\left\{\begin{pmatrix} a & b \\ b & a \end{pmatrix} : b = 2a,\ a \in \mathbb{R}\right\}$

   (b) The set of all functions $f(x)$ from $\mathbb{R}$ to $\mathbb{R}$ under the usual addition and scalar multiplication of functions satisfying $f(x+3) = f(x)$.

   (c) The set of vectors

$$\left\{\mathbf{v} \in \mathbb{R}^2 : \ \mathbf{v} = \begin{pmatrix} x \\ 0 \end{pmatrix} \text{ or } \mathbf{v} = \begin{pmatrix} 0 \\ y \end{pmatrix}, x \in \mathbb{R},\ y \in \mathbb{R}\right\}$$

4. Suppose that $B = \begin{pmatrix} 1 & -3 & -7 & 9 \\ 0 & 1 & 5 & -3 \\ 1 & -2 & -2 & 6 \end{pmatrix}$ and $\text{REF}(B) = \begin{pmatrix} 1 & -3 & -7 & 9 \\ 0 & 1 & 5 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and $\text{REF}(B^T) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

   (a) Find bases for the range of B and the cokernel of B.

   (b) Find $\dim(\ker(B))+\text{rank}(B)$.

   (c) Show that for an arbitrary $m \times n$ matrix C, $\dim(\ker(C))+\ \text{rank}(C) = n$.

5. Suppose that $\langle f, g\rangle = \displaystyle\int_{-1}^{+1} f(x)g(x)\,\mathrm{e}^x\,dx$ (†) where f, g are functions in $C^0[-1,+1]$.

   (a) Write down the norm associated with the above inner product (†).

   (b) Write down the formula for the Cauchy-Schwarz Inequality for (†).

   (c) Let $f(x) = x$ and $g(x) = e^{-x}$. Verify the Cauchy-Schwarz inequality you wrote down in (a).

   (d) Is the following statement true or false (as usual justify your answer): *If the vectors* $\boldsymbol{v}$ *and* $\boldsymbol{w}$ *in a vector space* V *are orthogonal with respect to one inner product on* V*, then they will be orthogonal with respect to any other inner product on* V*.*

6. Suppose that $f(x), g(x) \in C^0[0,\pi]$ with associated inner product $\langle f, g\rangle = \displaystyle\int_0^{\pi} f(x)g(x)\,dx$. You can assume that $\langle \sin(nx), \sin(mx)\rangle = 0$ if $n \neq m$ where m, n are non-negative integers and that $\langle \sin(nx), \sin(nx)\rangle = \dfrac{\pi}{2}$.

   (a) Suppose $f(x) = c_1 \sin(x) + c_2 \sin(2x) + c_3 \sin(3x) + ...$ Find formulae for the coefficients c_1, c_2, c_3, ... in terms of inner products.

   (b) Show that the functions $\sin(x), \sin(2x), \sin(3x), \cdots$ are linearly independent.