Solutions

UNIVERSITY OF CALGARY

UNIVERSITY OF CALGARY
FACULTY OF SCIENCE
DEPARTMENT OF MATHEMATICS & STATISTICS

MIDTERM

LINEAR METHODS I – MATH 211
ALL SECTIONS – WINTER 2017

Exam version: 0

Date: March 9, 2017 **Time: 2 hours**

SECTION NUMBER	LAB NUMBER	INSTRUCTOR NAME	STUDENT I.D. NUMBER	FIRST NAME	LAST NAME

EXAMINATION RULES

1. This is a closed book examination.
2. No aids are allowed for this examination.
3. Answer Part A on the scantron sheet and Part B on the examination paper.
4. Scantron sheets must be filled in during the exam time period. No additional time will be granted to fill in the scantron form.
5. The use of personal electronic or communication devices is prohibited.
6. A University of Calgary Student ID card is required to write the Final Examination and could be requested for midterm examinations. If adequate ID isn't presented, the student must complete an Identification Form.
7. Students late in arriving will not be permitted after one-half hour of the examination time has passed.
8. No student will be permitted to leave the examination room during the first 30 minutes, nor during the last 15 minutes of the examination. Students must stop writing and hand in their exam immediately when time expires.
9. All inquiries and requests must be addressed to the exam supervisor.
10. Students are strictly cautioned against:
    (a) communicating to other students;
    (b) leaving answer papers exposed to view;
    (c) attempting to read other students' examination papers.
11. During the final examination, if a student becomes ill or receives word of domestic affliction, the student must report to the Invigilator, hand in the unfinished paper and request that it be cancelled. If ill, the student must report immediately to a physician/counselor for a medical note to support a deferred examination application.
12. Once the examination has been handed in for marking, a student cannot request that the examination be cancelled. Retroactive withdrawals from the course will be denied.
13. Failure to comply with these regulations will result in rejection of the examination paper.

Question	Total Marks	Actual Marks
B1	5	
B2	5	
Total	10	

Part A: Mark your answers on the scantron sheet provided. **Exam version: 0**

Each question is worth 2 points.

1. Which of the following options constitute the general solution of the following system of linear equations:

$$\begin{array}{rcrcrcl} x & + & y & - & z & = & 0 \\ x & + & y & + & z & = & 4 \end{array}$$

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 1 & 1 & 1 & 4 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 1 & 0 & +2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

(a) $x = s$, $y = s$, $z = 4$

(b) $x = -s$, $y = -s$, $z = 2$

(c) $x = 4 - s$, $y = s$, $z = 2$

(d) $x = 2 - s$, $y = s$, $z = 2$

(e) None of the above.

$x = 2 - s$

$y = s$

$z = +2$

2. If a homogeneous system of m linear equations in n variables $AX = 0$ has infinitely many solutions, then what can we generally say about the $m \times n$ coefficient matrix A?

(a) It always has a row of zeros.

(b) It cannot be a square matrix.

(c) Its reduced row echelon form must have a row of zeros.

(d) Its reduced row echelon form certainly is not the identity matrix.

(e) None of the above statements are always true.

3. Let $A = \begin{bmatrix} 2 & 0 & 1 & 1 \\ 1 & -1 & 3 & 0 \\ 7 & 0 & 1 & 4 \\ 4 & 0 & 3 & 0 \end{bmatrix}$. Which of the following statements is **incorrect**?

(a) The reduced row-echelon form of A is the identity matrix I_4.

(b) rank $A = 4$.

(c) det $A = -9$.

(d) A is invertible.

(e) All the statements above are correct.

this implies all above statements are correct. $\Leftarrow$

det A = (along column 2)

$$= 0 - 1 \begin{vmatrix} 2 & 1 & 1 \\ 7 & 1 & 4 \\ 4 & 3 & 0 \end{vmatrix} + 0 + 0$$

$$= -1[4(4-1) - 3(8-7) + 0] =$$

$$= -(12-3) = -9$$

Exam version: 0

4. Consider an $n \times n$ matrix A satisfying the equation:

$$A^3 = (I - A)(I + A)$$

Which of the following is a correct expression for its inverse A^{-1}?

(a) A^2.
(b) $A^2 + A$.
(c) $A^2 - A$.
(d) $A + I$.
(e) None of the above.

$A^3 = I + \cancel{A} - \cancel{A} - A^2$

$A^3 + A^2 = I$

$A(\underbrace{A^2 + A}_{A^{-1}}) = I$

5. Consider the matrix $A = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix}$. Which statement is **correct**?

$\det A = -1(0-4) - 2(4-2)$
$= 4 - 4 = 0$

(a) There exists a matrix C such that $AC = I_3$.
(b) A is invertible.
(c) The system of linear equations $AX = B$ has a unique solution for any B.
(d) The system of linear equations $AX = B$ has no solution for some B.
(e) None of the above.

6. Let A, B, and C be 3×3 matrices with det $(A^{-1}) = -1$, det $(B) = -2$, and det $(C) = \frac{1}{2}$. Then

$$\det\left(\frac{1}{2}A^3 \text{adj}\,(B)(C^T)^2\right)$$

is equal to:

$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times (-1)^3 \times (-2)\cancel{(-2)} \times \cancel{\frac{1}{2}} \times \cancel{\frac{1}{2}}$

$= \frac{-1}{8}$

(a) 8.
(b) $-\frac{1}{2}$.
(c) $\frac{1}{4}$.
(d) $-\frac{1}{8}$.
(e) None of the above.

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7. Consider the matrix $A = \begin{bmatrix} 1 & c & 0 \\ 2 & 0 & c \\ c & 1 & 4 \end{bmatrix}$. If A is not invertible, then:

   (a) $c = 0$.
   (b) c is 0 or 3 or 1.
   (c) c is 0 or -3 or -1.
   (d) c is 0 or 3 or -3.
   (e) c is 1 or -1.

$\det A = 1(0-c) - c(8-c^2) + 0$
$= -c - c(8-c^2)$
$= c(-1-(8-c^2))$
$= c(c^2-9) = c(c+3)(c-3)$

A not invertible $\Leftrightarrow$ $\det A = 0$
$\Leftrightarrow c = 0, 3, -3$

8. Find A if

$$(A^{-1} - 2I)^{-1} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}.$$

   (a) $A = \begin{bmatrix} 3/2 & 2 \\ 1 & 3/2 \end{bmatrix}$.
   (b) $A = \begin{bmatrix} 3 & 2 \\ 1 & 3 \end{bmatrix}$.
   (c) $A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$.
   (d) $A = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$.
   (e) None of the above.

$A^{-1} - 2I = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & +2 \\ +1 & -1 \end{bmatrix}$

$A^{-1} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} -1 & 2 \\ +1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$

$A = \begin{bmatrix} -1 & +2 \\ +1 & -1 \end{bmatrix}$

9. Which of the following statements is always true?

   (a) The sum of two invertible $n \times n$ matrices is invertible.
   (b) The difference of two invertible $n \times n$ matrices is invertible.
   (c) A scalar multiple of an invertible $n \times n$ matrix is invertible.
   (d) The product of two invertible $n \times n$ matrices is invertible.
   (e) None of the above.

$(AB)^{-1} = B^{-1}A^{-1}$

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10. Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 3 & 0 & 3 \end{bmatrix}$ and let R be the reduced row-echelon form of the matrix A.

Find a matrix U such that $R = UA$.

(a) $U = \begin{bmatrix} 0 & 1/3 \\ 1/3 & -1/3 \end{bmatrix}$.

(b) $U = \begin{bmatrix} 0 & 1/3 \\ 1 & -1/3 \end{bmatrix}$.

(c) $U = \begin{bmatrix} 0 & 1/3 \\ -1/3 & 1 \end{bmatrix}$.

(d) $U = \begin{bmatrix} 1 & 0 \\ 3 & -1 \end{bmatrix}$.

(e) None of the above.

$-3R_1+R_2$ $\begin{bmatrix} 1 & 1 & 0 \\ 0 & -3 & 3 \end{bmatrix}$ $E_1 = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$

$R_2/-3$ $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$ $E_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1/3 \end{bmatrix}$

$-R_2+R_1$ $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$ $E_3 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

$R = UA$

$U = E_3E_2E_1 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1/3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 1/3 \\ 0 & -1/3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1/3 \\ 1 & -1/3 \end{bmatrix}$

11. Which of the following point does not lie on the line through $A(2, 3, -1)$ and $B(5, 1, -1)$?

(a) $P(8, -1, -1)$.

(b) $P(2, 3, -1)$.

(c) $P(3, -2, -1)$.

(d) $P(-1, 5, -1)$.

(e) None of the above.

$\begin{cases} x = 2 + 3s \\ y = 3 - 2s \\ z = -1 \end{cases}$

$x = 3 \Rightarrow s = 1/3 \Rightarrow y \neq -2$

12. Let A be a 3×3 matrix. The reduced row echelon form of A is the identity matrix, and is obtained by performing the following three elementary row operations in order:

(i) First multiply row 1 by $1/3$.

(ii) Then add -2 times row 1 to row 2.

(iii) Then finally interchange rows 2 and 3.

Then what is det (A)?

(a) det $(A) = 3$.

(b) det $(A) = -3$.

(c) det $(A) = 1/3$.

(d) det $(A) = -1/3$.

(e) None of the above.

$I = E_3E_2E_1A$

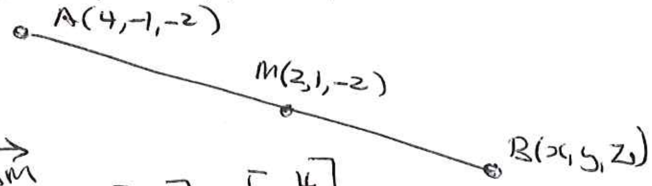
$\det I = \det E_3 \det E_2 \det E_1 \det A$

$1 = (-1)(1)(\frac{1}{3}) \det A$

$\therefore \det A = -3$

Exam version: 0

13. Find the point B such that $M(2,1,-2)$ is midway between points $A(4,-1,-2)$ and B.



(a) $B(3,0,-2)$.
(b) $B(0,3,-2)$.
(c) $B(0,3,0)$.
(d) $B(0,0,-2)$.
(e) None of the above.

$$\vec{OB} = \vec{OA} + \vec{AB} = \vec{OA} + 2\vec{AM} = \begin{bmatrix}4\\-1\\-2\end{bmatrix} + 2\begin{bmatrix}-2\\+2\\0\end{bmatrix} = \begin{bmatrix}4\\-1\\-2\end{bmatrix} + \begin{bmatrix}-4\\+4\\0\end{bmatrix} = \begin{bmatrix}0\\+3\\-2\end{bmatrix}$$

14. Consider the following two lines L_1 and L_2 in $\mathbb{R}^3$:

$$L_1 : \begin{cases} x = -3 - 2t \\ y = 7 + 4t \\ z = 1 + t \end{cases} \qquad L_2 : \begin{cases} x = 1 + s \\ y = -1 - 2s \\ z = 2 + s \end{cases}$$

Which of the following statements is true?

(a) L_1 and L_2 are the same line.
(b) L_1 and L_2 are parallel but do not intersect.
(c) L_1 and L_2 are not parallel and do not intersect.
(d) L_1 and L_2 are not parallel and do intersect.
(e) None of the above.

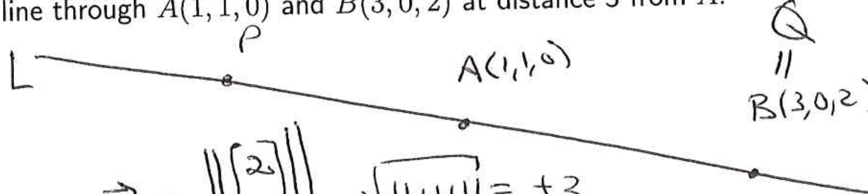
$$\begin{cases} -3-2t = 1+s \\ 7+4t = -1-2s \\ 1+t = 2+s \end{cases}$$

$$\begin{cases} s+2t = -4 \\ \cancel{2s+4t = -8} \\ s-t = -1 \end{cases}$$

$\Rightarrow 3t = -3$, so $t = -1$, $s = -2$

Unique point of intersection, so they intersect but cannot be parallel.

15. Find the two points P and Q on the line through $A(1,1,0)$ and $B(3,0,2)$ at distance 3 from A.



(a) $P(-1,0,2)$ and $Q(3,0,-2)$.
(b) $P(-1,2,-2)$ and $Q(3,0,-2)$.
(c) $P(-1,2,2)$ and $Q(-1,0,2)$.
(d) $P(-1,2,2)$ and $Q(-1,2,-2)$.
(e) None of the above.

(Typo) Both answers accepted

$$\|\vec{AB}\| = \left\|\begin{bmatrix}2\\-1\\2\end{bmatrix}\right\| = \sqrt{4+1+4} = +3$$

so $\vec{OQ} = \vec{OA} + \vec{AB} = \vec{OB}$ ie $Q = B$

$$\vec{OP} = \vec{OA} - \vec{AB} = \begin{bmatrix}1\\1\\0\end{bmatrix} - \begin{bmatrix}2\\-1\\2\end{bmatrix} = \begin{bmatrix}-1\\2\\-2\end{bmatrix}, \text{ ie } P(-1,2,-2)$$

Part B: Provide your full solution on these pages. **Exam version: 0**

Question 1 (5 points) With full explanation, determine the quadratic $y = ax^2 + bx + c$ whose graph contains the points $(0,1)$, $(1,2)$ and $(2,4)$.

Write a clear and complete solution, and provide your final answer in the box

3 equat

$$\left|\begin{array}{l} 0a + 0b + c = 1 \\ a + b + c = 2 \\ 4a + 2b + c = 4 \end{array}\right.$$

Aug

$$\left[\begin{array}{cccc} 1 & 1 & 1 & 2 \\ 4 & 2 & 1 & 4 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\downarrow \left[\begin{array}{cccc} 1 & 1 & 1 & 2 \\ 0 & -2 & -3 & -4 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\left[\begin{array}{cccc} 1 & 1 & 0 & 1 \\ 0 & -2 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\left[\begin{array}{cccc} 1 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 1 \end{array}\right] \Leftarrow \begin{array}{l} a = 1/2 \\ b = 1/2 \\ c = 1 \end{array}$$

ANSWER: $y = \frac{x^2}{2} + \frac{x}{2} + 1$

Exam version: 0

Question 2 (5 points) Consider the following augmented matrix of a system of linear equations, where a can be any real number:

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 1 \\ 1 & a^2-1 & 5 & 2 \\ 1 & 0 & a+2 & a+1 \end{array}\right].$$

Find all values of a such that the system of linear equations has either no solution, a unique solution, or infinitely many solutions.

Write a clear and complete solution, and provide your final answer in the box below.

$-R_1 + R_2$
$-R_1 + R_3$

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 1 \\ 0 & a^2-1 & 2 & 1 \\ 0 & 0 & a-1 & a \end{array}\right]$$

① If $a = +1$ $\Rightarrow$ $\left[\begin{array}{ccc|c} 1 & 0 & 3 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right]$ No solution

② If $a = -1$ $\Rightarrow$ $\left[\begin{array}{ccc|c} 1 & 0 & 3 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & -2 & -1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 3 & 1 \\ 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 0 \end{array}\right]$

Infinitely many solutions $\begin{cases} x = -1/2 \\ y = s \\ z = 1/2 \end{cases}$

③ if $a \neq \pm 1$, unique solution

ANSWER:	No solution:	$a = +1$
	Unique solution:	$a \neq \pm 1$
	Infinitely many solutions:	$a = -1$

Exam version: ~~0~~ 1

Part B: Provide your full solution on these pages.

Question 1 (5 points) With full explanation, determine the quadratic $y = ax^2 + bx + c$ whose graph contains the points $(0, -2)$, $(1, 0)$ and $(2, 4)$.

Write a clear and complete solution, and provide your final answer in the box below.

We obtain 3 linear equations in variables a, b, c

$$\begin{cases} 0a + 0b + c = -2 \\ a + b + c = 0 \\ 4a + 2b + c = 4 \end{cases}$$

Augmented matrix

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 4 & 2 & 1 & 4 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

$-$Row 3 + Row 1, $-$Row 3 + Row 2

$$\begin{bmatrix} 1 & 1 & 0 & 2 \\ 4 & 2 & 0 & 6 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

-4Row 1 + Row 2

$$\begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

$-\frac{1}{2}$Row 2

$$\begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

$-$Row 2 + Row 1

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix} \Longrightarrow \begin{matrix} a = 1 \\ b = 1 \\ c = -2 \end{matrix}$$

ANSWER: $y = x^2 + x - 2$

Exam version: ~~0~~ 1

Question 2 (5 points) Consider the following augmented matrix of a system of linear equations, where a can be any real number:

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 1 & a^2-4 & 4 & 2 \\ 1 & 0 & a & a+1 \end{array}\right].$$

Find all values of a such that the system of linear equations has either no solution, a unique solution, or infinitely many solutions.

Write a clear and complete solution, and provide your final answer in the box below.

$-$Row 1 + Row 2 , $-$Row 1 + Row 3

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & a^2-4 & 2 & 1 \\ 0 & 0 & a-2 & a \end{array}\right]$$

① If $a = +2 \Rightarrow \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} \Rightarrow$ NO SOLUTION

② If $a = -2 \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & -4 & -2 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$

Infinitely many solution $\begin{cases} x = 0 \\ y = s \\ z = 1/2 \end{cases}$

③ If $a \neq \pm 2$, unique solution

ANSWER:	No solution:	$a = +2$
	Unique solution:	$a \neq \pm 2$
	Infinitely many solutions:	$a = -2$

Part B: Provide your full solution on these pages. **Exam version:** ~~0~~ 2

Question 1 (5 points) With full explanation, determine th2 quadratic $y = ax^2 + bx + c$ whose graph contains the points $(0, -2)$, $(1, -2)$ and $(2, 0)$.

Write a clear and complete solution, and provide your final answer in the box below.

We obtain 3 linear equations in variables a, b, c:

$$\begin{cases} 0a + 0b + 0c = -2 \\ a + b + c = -2 \\ 4a + 2b + c = 0 \end{cases}$$

Augmented matrix

$$\begin{bmatrix} 1 & 1 & 1 & -2 \\ 4 & 2 & 1 & 0 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

$-$Row 3 + Row 1, $-$Row 3 + Row 2

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 4 & 2 & 0 & 2 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

-4 Row 1 + Row 2

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -2 & 0 & 2 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

$-\frac{1}{2}$ Row 2 $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$

$-$ Row 2 + Row 1

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 \end{bmatrix} \Rightarrow \begin{matrix} a = 1 \\ b = -1 \\ c = -2 \end{matrix}$$

ANSWER: $y = x^2 - x - 2$.

Exam version: ~~0~~ 2

Question 2 (5 points) Consider the following augmented matrix of a system of linear equations, where a can be any real number:

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & -1 \\ 1 & a^2-4 & 4 & 0 \\ 1 & 0 & a+4 & a-1 \end{array}\right].$$

Find all values of a such that the system of linear equations has either no solution, a unique solution, or infinitely many solutions.

Write a clear and complete solution, and provide your final answer in the box below.

$-$Row 1 + Row 2, $-$Row 1 + Row 3

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & -1 \\ 0 & a^2-4 & 2 & 1 \\ 0 & 0 & a+2 & a \end{array}\right]$$

① if $a = -2 \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & -1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & -2 \end{array}\right]$ No solution

② if $a = +2 \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & -1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 4 & 2 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & -1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$

Infinitely many solution $\begin{cases} x = -2 \\ y = s \\ z = 1/2 \end{cases}$

③ if $a \neq \pm 2$, Unique solution

ANSWER:	No solution:	$a = -2$.
	Unique solution:	$a \neq \pm 2$
	Infinitely many solutions:	$a = +2$.