UNIVERSITY OF CALGARY

UNIVERSITY OF CALGARY
FACULTY OF SCIENCE
DEPARTMENT OF MATHEMATICS & STATISTICS

MIDTERM

LINEAR METHODS I – MATH 211
ALL SECTIONS – FALL 2016

Exam version: 0

Date: October 21, 2016 Solutions **Time: 2 hours**

SECTION NUMBER	LAB NUMBER	STUDENT I.D. NUMBER	FIRST NAME	LAST NAME

EXAMINATION RULES

1. This is a closed book examination.
2. No aids are allowed for this examination.
3. Answer Part A on the scantron sheet and Part B on the examination paper.
4. Scantron sheets must be filled in during the exam time period. No additional time will be granted to fill in the scantron form.
5. The use of personal electronic or communication devices is prohibited.
6. A University of Calgary Student ID card is required to write the Final Examination and could be requested for midterm examinations. If adequate ID isn't presented, the student must complete an Identification Form.
7. Students late in arriving will not be permitted after one-half hour of the examination time has passed.
8. No student will be permitted to leave the examination room during the first 30 minutes, nor during the last 15 minutes of the examination. Students must stop writing and hand in their exam immediately when time expires.
9. All inquiries and requests must be addressed to the exam supervisor.
10. Students are strictly cautioned against:
    (a) communicating to other students;
    (b) leaving answer papers exposed to view;
    (c) attempting to read other students' examination papers.
11. During the final examination, if a student becomes ill or receives word of domestic affliction, the student must report to the Invigilator, hand in the unfinished paper and request that it be cancelled. If ill, the student must report immediately to a physician/counselor for a medical note to support a deferred examination application.
12. Once the examination has been handed in for marking, a student cannot request that the examination be cancelled. Retroactive withdrawals from the course will be denied.
13. Failure to comply with these regulations will result in rejection of the examination paper.

Question	Total Marks	Actual Marks
B1	5	
B2	5	

Part A: Mark your answers on the scantron sheet provided. **Exam version: 0**

Each question is worth 2 points.

1. What can you say about the general solution of the following system:

$$\begin{array}{rcrcrcl} x & - & y & + & 2z & = & 1 \\ -2x & + & 2y & - & 4z & = & 2 \end{array}$$

   (a) It has a unique solution.
   (b) It has no solution.
   (c) The general solution has exactly one parameter.
   (d) The general solution has exactly two parameters.
   (e) The general solution has exactly three parameters.

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ -2 & 2 & -4 & 2 \end{array}\right] \xrightarrow{r_2+2r_1} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 0 & 0 & 4 \end{array}\right]$$

2. The rank of the matrix $\begin{bmatrix} 1 & -3 & 2 \\ 1 & 0 & -1 \\ 1 & -4 & 3 \\ 1 & 1 & -2 \end{bmatrix}$ is

   (a) 0.
   (b) 1.
   (c) 2.
   (d) 3.
   (e) 4.

$$\begin{bmatrix} 1 & -3 & 2 \\ 1 & 0 & -1 \\ 1 & -4 & 3 \\ 1 & 1 & -2 \end{bmatrix} \xrightarrow[\substack{r_3-r_1 \\ r_4-r_1}]{r_2-r_1} \begin{bmatrix} 1 & -3 & 2 \\ 0 & 3 & -3 \\ 0 & -1 & 1 \\ 0 & 4 & -4 \end{bmatrix} \xrightarrow{r_2 \leftrightarrow r_3} \begin{bmatrix} 1 & -3 & 2 \\ 0 & -1 & 1 \\ 0 & 3 & -3 \\ 0 & 4 & -4 \end{bmatrix}$$

$$\xrightarrow[\substack{r_3+3r_2 \\ r_4+4r_2}]{} \begin{bmatrix} 1 & -3 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -3 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

2 leading

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3. Consider the following augmented matrix in which $*$ denotes an arbitrary number and ■ denotes a **nonzero** number.

$$\left[\begin{array}{cccc|c} \blacksquare & * & * & * & \blacksquare \\ 0 & 0 & \blacksquare & 0 & \blacksquare \\ 0 & 0 & \blacksquare & 0 & 0 \end{array}\right]$$

can not be satisfy with $X_3=0$

$X_3=0$

What can you say about the corresponding system of linear equations?

(a) It has no solution.
(b) It has a unique solution.
(c) The general solution has exactly one parameter.
(d) The general solution has exactly two parameters.
(e) There is insufficient information to determine the answer.

4. Consider the following augmented matrix of a system of linear equations, where a is any number:

$$\left[\begin{array}{ccc|c} 1 & 5 & 0 & 1 \\ 0 & 0 & 0 & a^2 \\ 0 & 0 & a-1 & 1 \end{array}\right].$$

Which of the following statements is true?

(a) For every a, the system has a unique solution.
(b) For every a, the system has no solution.
(c) The system has a unique solution when $a = 0$.
(d) The system has infinitely many solutions when $a = 0$.
(e) The system has a unique solution when $a = 1$.

no leading 1 on this column

$$\left[\begin{array}{ccc|c} 1 & 5 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{array}\right]$$

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5. Let A be the augmented matrix and C the coefficient matrix for a system of m linear equations in n variables. If $\text{rank}(A) = \text{rank}(C) = n$, which of the following statements must be true?

(a) The system has no solution.
(b) The system has a unique solution.
(c) The general solution must have at exactly one parameter.
(d) The general solution has at least one parameter.
(e) There is insufficient information to determine the answer.

6. Find all real values of a so that the matrix $A = \begin{bmatrix} a+1 & 0 & -3 \\ 0 & a+1 & 0 \\ 4 & 0 & a-6 \end{bmatrix}$ is invertible.

(a) Any value of a except -1, 2 and 3.
(b) Only -1, 2 and 3.
(c) Any value of a except -1 and 6.
(d) Only -1 and 6.
(e) The matrix A is invertible for any value of a.

$$\det(A) = (a+1)\begin{vmatrix} a+1 & -3 \\ 4 & a-6 \end{vmatrix} = (a+1)[(a+1)(a-6)+12]$$

cofactor expension along the second row

$$= (a+1)[a^2-5a+6]$$

$$= (a+1)(a-2)(a-3)$$

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7. Find A if

$$\left(A^{-1} + 3\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\right)^{-1} = 4A.$$

(a) $A = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$.

(b) $A = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.

(c) $A = \frac{1}{5}\begin{bmatrix} 0 & -1 \\ -1 & 2 \end{bmatrix}$.

(d) $A = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$.

(e) $A = \frac{1}{4}\begin{bmatrix} 0 & -1 \\ -1 & 1 \end{bmatrix}$.

$$A^{-1} + 3\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \frac{1}{4}A^{-1}$$

$$\frac{3}{4}A^{-1} = -3\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$A^{-1} = -4\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$A = -\frac{1}{4}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \frac{1}{4}\begin{bmatrix} 0 & -1 \\ -1 & 1 \end{bmatrix}$$

8. Find the $(2,3)$-entry of the product AB given

$$A = \begin{bmatrix} 1 & 4 & -2 \\ 1 & -2 & 1 \\ 2 & 1 & -9 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 17 & 6 & 4 \\ 10 & -9 & 2 \\ 3 & 0 & 3 \end{bmatrix}.$$

(a) 3.

(b) 2.

(c) 1.

(d) 0.

(e) -47.

$$(AB)_{2,3} = 1\cdot 4 + (-2)\cdot 2 + 1\cdot 3 = 3$$

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9. A square 2×2 matrix A is called *nilpotent* if $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Which of the following matrices are nilpotent?

$$B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

(a) Only B.
(b) Only C.
(c) Only B and C.
(d) Only B and D.
(e) all of B, C and D.

$B^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$C^2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$D^2 = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

10. Given a square matrix A, which of the following statements is **not** equivalent to A being invertible?

(a) The matrix A can be written as a product of elementary matrices.
(b) The reduced row-echelon form of A is I.
(c) The system of linear equations $AX = B$ has a unique solution for any B.
(d) The system of linear equations $AX = 0$ has infinitely many solutions.
(e) All the above statements are equivalent to A being invertible.

Please note that in Version 0 of the test this question was slightly different, and the correct answer of ver 0 of the Midterm is (e)

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11. Let A be a 2×9 matrix, and let B be the matrix obtained from A by performing the following two elementary row operations in order:

(i) add -5 times row 2 to row 1.

(ii) interchange row 1 and row 2;

Then $B = UA$ where U is which one of the following matrices?

(a) $\begin{bmatrix} 0 & 1 \\ 1 & -5 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 \\ -5 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} -5 & 1 \\ 1 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 5 \\ 1 & 1 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & -5 \\ 0 & 1 \end{bmatrix}$

$$A \xrightarrow{E_1} E_1A \xrightarrow{E_2} E_2(E_1A) = B$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{r_1 - 5r_2} \begin{bmatrix} 1 & -5 \\ 0 & 1 \end{bmatrix} = E_1$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{r_1 \leftrightarrow r_2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = E_2$$

$$U = E_2E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & -5 \\ 0 & 1 \end{bmatrix} = \cancel{\begin{bmatrix} 0 & 1 \\ 1 & -5 \end{bmatrix}}$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & -5 \end{bmatrix}$$

12. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}$. Then the $(2,3)$ −entry of A^{-1} is

$$A^{-1} = \frac{1}{det(A)} adj(A)$$

(a) $2.$

(b) $-2.$

(c) $\frac{1}{2}.$

(d) $-\frac{1}{2}.$

(e) $-1.$

$$(A^{-1})_{23} = \frac{1}{det(A)}(adj(A))_{2,3} = \frac{1}{det(A)}\left[(cof(A))^T\right]_{(2,3)}$$

$$\Rightarrow (A^{-1})_{2,3} = \frac{1}{det(A)} cof(A)_{(3,2)} = -\frac{1}{2}$$

$$det(A) = 1 \cdot \begin{vmatrix} 1 & 1 \\ 2 & 4 \end{vmatrix} = 4-2 = 2 \qquad cof(A)_{3,2} = (-1)^{2+3}\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = -1$$

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13. Find the matrix C so that $\operatorname{adj}(C) = \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix}$.

    (a) $C = \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix}$.

    (b) $C = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix}$.

    (c) $C = 2\begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix}$.

    (d) $C = \begin{bmatrix} 4 & -1 \\ -2 & 1 \end{bmatrix}$.

    (e) $C = \begin{bmatrix} -1 & 2 \\ 1 & -4 \end{bmatrix}$.

14. Let A, B and C be 2×2 matrices, with det $(A) = -1$, det $(B) = 2$, and det $(C) = -6$. Which of the following values represents det $\left(A^{-1}C^2(3B^T)^{-1}A^2\right)$?

    (a) -216.

    (b) -162.

    (c) -54.

    (d) -6.

    (e) -2.

$det(A^{-1}C^2(3B^T)^{-1}A^2) = det(A^{-1})det(C^2)det[(3B^T)^{-1}]$

$\times det$

$= det(A)(det(C))^2 det(\frac{1}{3}(B^T)^{-1}) = (-1)(-6)^2 \cdot (\frac{1}{3})^2 \cdot \frac{1}{}$

$= -36 \cdot \frac{1}{9} \cdot \frac{1}{2} = -2$

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15. Let A and B be $n \times n$ matrices. Choose the statement that is **always** true.

    (a) If $A^T = A$ and $B^T = B$, then $(AB)^T = AB$.

    (b) If $A^{-1} = B$ then $AB = BA$.

    (c) If A and B are invertible, then $(A+B)^{-1} = A^{-1} + B^{-1}$.

    (d) If $A + B$ is invertible, then A is invertible and B is invertible.

    (e) $(A+B)(A-B) = A^2 - B^2$.

Part B: Provide your full solution on these pages. **Exam version: 0**

Question 1 (5 points) *Consider a system of linear equations in the form $AX = B$, for which the reduced row-echelon form of the augmented matrix is:*

$x_2 = s \quad x_5 = t$

$$\left[\begin{array}{ccccc|c} 1 & -2 & 0 & 0 & 5 & -7 \\ 0 & 0 & 1 & 0 & -4 & 6 \\ 0 & 0 & 0 & 1 & 3 & -8 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Write a clear and complete solution, and provide your final answers in the boxes below:

(a) **[1 mark]** How many solution(s) the the system have?

infinitely many solutions

(b) **[2 marks]** Find the general solution and express it in a **vector** form.

$x_1 = 2s - 5t - 7$

$x_2 = s$

$x_3 = 4t + 6$

$x_4 = -3t - 8$

$x_5 = t$

$$\Rightarrow \vec{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2s \\ s \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -5t \\ 0 \\ 4t \\ -3t \\ t \end{bmatrix} + \begin{bmatrix} -7 \\ 0 \\ 6 \\ -8 \\ 0 \end{bmatrix}$$

$$X = s\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -5 \\ 0 \\ 4 \\ -3 \\ 1 \end{bmatrix} + \begin{bmatrix} -7 \\ 0 \\ 6 \\ -8 \\ 0 \end{bmatrix}$$

(c) **[2 marks]** Find a set of basic solutions to the associated homogeneous system $AX = 0$.

$$\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 4 \\ -3 \\ 1 \end{bmatrix}$$

(a)	**(b)**	**(c)** Basic Solutions:
infinitely many solutions	$X = s\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -5 \\ 0 \\ 4 \\ -3 \\ 1 \end{bmatrix} + \begin{bmatrix} -7 \\ 0 \\ 6 \\ -8 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 4 \\ -3 \\ 1 \end{bmatrix}$

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Question 2 (5 points)

Write a clear and complete solution, and provide your final answer in the box below:

(a) **[3 marks]** Use the matrix inversion algorithm to find A^{-1} when $A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -2 \\ -1 & -1 & 3 \end{bmatrix}$.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ -1 & -1 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow{r_3 + r_1} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & -2 & 5 & 1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[r_3 + 2r_2]{r_1 + r_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 2 & 1 \end{array}\right] \xrightarrow{r_2 + 2r_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 2 & 5 & 2 \\ 0 & 0 & 1 & 1 & 2 & 1 \end{array}\right]$$

$$\Longrightarrow \quad A^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 5 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

(b) **[2 marks]** Let $M = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & -1 \\ -1 & -1 & -3 \end{bmatrix}$ and let $B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

Solve the linear system $MX = B$ for X, using $M^{-1} = \begin{bmatrix} 1 & -2 & 1 \\ 2 & -1 & 1 \\ -1 & 1 & -1 \end{bmatrix}$

$$X = M^{-1}B = \begin{bmatrix} 1 & -2 & 1 \\ 2 & -1 & 1 \\ -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1-4+3 \\ 2-2+3 \\ -1+2-3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix}$$

$$\boxed{A^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 5 & 2 \\ 1 & 2 & 1 \end{bmatrix} \qquad X = \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix}}$$

End of Examination

scrap paper

End of Examination