SoLuTions

UNIVERSITY OF CALGARY

Faculty of Science
Department of Mathematics & Statistics

MATH 211 – LINEAR METHODS I – WINTER 2014

MIDTERM 1

February 6, 2014
Exam version: 0

COURSE INFORMATION AND STUDENT IDENTIFICATION

INSTRUCTOR NAME	SECTION NUMBER	LAB NUMBER	STUDENT I.D. NUMBER	FIRST NAME	LAST NAME

EXAMINATION RULES

1. No Calculators, electronic equipment, or paper material other than this examination and scantron sheet allowed.
2. Use the back of the previous page for rough work or calculations.
3. Students arriving late will not normally be admitted after one-half hour of the examination time has passed.
4. No candidate will be permitted to leave the examination room until one hour has elapsed after the opening of the examination.
5. All enquiries and requests must be addressed to supervisors only.
6. Candidates are strictly cautioned against:
   (a) speaking to other candidates or communicating with them under any circumstances whatsoever;
   (b) bringing into the examination room any textbook, notebook or memoranda not authorized by the examiner;
   (c) making use of calculators and/or portable computing machines not authorized by the instructor;
   (d) leaving answer papers exposed to view;
   (e) attempting to read other students' examination papers.

   The penalty for violation of these rules is suspension or expulsion or such other penalty as may be determined.
7. Discarded matter is to be struck out and not removed by mutilation of the examination answer book.
8. Candidates are cautioned against writing in their answer books any matter extraneous to the actual answering of the question set.
9. A candidate must report to a supervisor before leaving the examination room.
10. Answer books must be handed to the supervisor-in-charge promptly when the signal is given. Failure to comply with this regulation will be cause for rejection of an answer paper.

Part A: Mark your answers on the scantron sheet provided. **Exam version: 0**

Each question is worth 2 points.

1. What can you say about the following system of linear equations?

$$\begin{array}{rcrcrcr} x & - & 2y & + & 3z & = & -1 \\ 2x & + & 5y & + & 6z & = & 4 \end{array}$$

$$\begin{bmatrix} 1 & -2 & 3 & -1 \\ 2 & 5 & 6 & 4 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} 1 & -2 & 3 & -1 \\ 0 & 9 & 0 & 6 \end{bmatrix}$$

   (a) It has no solution.
   (b) It has a unique solution.
   (c) The general solution has exactly one parameter.
   (d) The general solution has exactly two parameters.
   (e) None of the above.

2. What can you say about the following system of linear equations?

$$\begin{array}{rcrcrcrl} -x_1 & + & 2x_2 & + & 3x_3 & + & x_4 = & 0 \\ 2x_1 & - & 4x_2 & - & 6x_3 & - & x_4 = & 0 \end{array}$$

$$\begin{bmatrix} -1 & 2 & 3 & 1 & 0 \\ 2 & -4 & -6 & -1 & 0 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} -1 & 2 & 3 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

   (a) It has no solution.
   (b) It has a unique solution.
   (c) The general solution involves exactly one parameter.
   (d) The general solution involves exactly two parameters.
   (e) The general solution involves exactly three parameters.

3. Which of the following options is the general solution of the following system of linear equations?

$$\begin{array}{rcrcrcr} x & - & y & - & z & = & 0 \\ x & + & y & + & z & = & 4 \end{array}$$

$$\begin{bmatrix} 1 & -1 & -1 & 0 \\ 1 & 1 & 1 & 4 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 2 & 2 & 4 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 2 \end{bmatrix}$$

   (a) $x = 2,\ y = 0,\ z = 2$
   (b) $x = 2,\ y = 1,\ z = 1$
   (c) $x = 2,\ y = 2 - s,\ z = s$
   (d) $x = 2,\ y = 2 + s,\ z = s$
   (e) None of the above.

$x = 2$
$y = 2-s$
$z = s$

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4. Consider the following homogeneous system of linear equations:

$$\begin{array}{rcrcrcrcrcl} x_1 & + & 2x_2 & + & 4x_3 & + & x_4 & - & 3x_5 & = & 0 \\ x_1 & + & 3x_2 & + & 4x_3 & + & x_4 & - & 3x_5 & = & 0 \\ x_1 & + & 2x_2 & + & 4x_3 & + & 2x_4 & - & 6x_5 & = & 0 \end{array}$$

Every solution $[x_1 \;\; x_2 \;\; x_3 \;\; x_4 \;\; x_5]^T$ of the above system is a linear combination of the following matrices:

$$\begin{bmatrix} 1 & 2 & 4 & 1 & -3 & 0 \\ 1 & 3 & 4 & 1 & -3 & 0 \\ 1 & 2 & 4 & 2 & -6 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 4 & 1 & -3 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -3 & 0 \end{bmatrix}$$

(a) $[-4 \;\; 0 \;\; 1 \;\; -6 \;\; -2]^T$.

(b) $[8 \;\; 0 \;\; -2 \;\; 3 \;\; 1]^T$.

(c) $[-4 \;\; 0 \;\; 1 \;\; 0 \;\; 0]^T$ and $[-8 \;\; 0 \;\; 2 \;\; 0 \;\; 0]^T$. $\to \begin{bmatrix} 1 & 0 & 4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -3 & 0 \end{bmatrix}$

(d) $[4 \;\; 0 \;\; -1 \;\; 0 \;\; 0]^T$ and $[0 \;\; 0 \;\; 0 \;\; 3 \;\; 1]^T$.

(e) None of the above.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -4s \\ 0 \\ s \\ 3t \\ t \end{bmatrix} = s\begin{bmatrix} -4 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ 0 \\ 3 \\ 1 \end{bmatrix}$$

5. Let C be the coefficient matrix for a system of m linear equations in n variables. Which of the following statements is always true?

(a) If the rank of C is less than n, then the system has infinitely many solutions.

(b) If the rank of C is equal to m, then the system is consistent.

(c) If the rank of C is less than m, then the system has infinitely many solutions.

(d) If the rank of C is equal to n, then the system has a unique solution.

(e) All the above statements are false.

6. Consider the following augmented matrix of a system of linear equations, where a is any number:

$$\left[\begin{array}{ccc|c} 1 & 5 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & a & 6 \end{array}\right]. \to \begin{bmatrix} 1 & 5 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 6-2a \end{bmatrix}$$

Which of the following statements is true?

(a) The system has a unique solution when $a = 3$.

(b) The system has no solution when $a = 3$.

(c) The system has infinitely many solutions when $a = 3$.

(d) The system has infinitely many solutions when $a \neq 3$.

(e) None of the above.

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7. Let $X_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, $X_2 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$, $W = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}$, $Y = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, and $Z = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

   Which of the following statements is true?

   (a) Y is a linear combination of X_1 and X_2, but W and Z are not.
   (b) W and Z are linear combinations of X_1 and X_2, but Y is not.
   (c) W and Y are linear combinations of X_1 and X_2, but Z is not.
   (d) Y is a linear combination of X_1 and X_2, but W and Z are not.
   (e) All the above statements are false.

$$\begin{array}{c} \begin{array}{ccccc} X_1 & X_2 & W & Y & Z \end{array} \\ \left[\begin{array}{cc|ccc} 1 & 1 & -1 & 1 & 0 \\ 2 & 1 & 0 & 0 & 1 \\ -1 & -2 & 3 & 0 & 1 \end{array}\right] \end{array}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 1 & -1 & 1 & 0 \\ 0 & -1 & 2 & -2 & 1 \\ 0 & -1 & 2 & 1 & 1 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 1 & -1 & 1 & 0 \\ 0 & -1 & 2 & -2 & 1 \\ 0 & 0 & 0 & 3 & 0 \end{bmatrix}$$

8. Let A and B be $n \times n$ matrices, and suppose U is an invertible matrix so that $B = UA$. Then which of the following statements is always true?

   (a) $A = BU^{-1}$.
   (b) $A = BU$.
   (c) $B = AU^{-1}$.
   (d) $B = AU$.
   (e) None of the above.

9. Find all values of a so that the matrix $A = \begin{bmatrix} 1 & 2 \\ a & 1 \end{bmatrix}$ is invertible.

   (a) $a = 0$.
   (b) $a \neq 0$.
   (c) $a = \frac{1}{2}$.
   (d) $a \neq \frac{1}{2}$.
   (e) None of the above.

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ a & 1 & 0 & 1 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1-2a & -a & 1 \end{bmatrix}$$

need $1 - 2a \neq 0$

$a \neq 1/2$

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10. Find A if

$$(A-2I)^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

$A - 2I = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$

$A = 2I + \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$

$= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

(a) $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$.
(b) $A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$.
(c) $A = \begin{bmatrix} -2 & 1 \\ 1 & -3 \end{bmatrix}$.
(d) $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.
(e) None of the above.

11. Compute the following matrix product and simplify your answer:

$$\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ a & -b \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

$= \begin{bmatrix} 2a & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = -2a$

(a) $\begin{bmatrix} a+b \end{bmatrix}$.
(b) $\begin{bmatrix} 2a \end{bmatrix}$.
(c) $\begin{bmatrix} -2a \end{bmatrix}$.
(d) $\begin{bmatrix} 2a+b \end{bmatrix}$.
(e) None of the above.

12. Let A and B be invertible matrices. Which of the following statements is always true?

(a) $(A+B)^{-1} = A^{-1} + B^{-1}$.
(b) $(A+B)^T = A^T + B^T$.
(c) $(AB)^{-1} = A^{-1}B^{-1}$.
(d) $(AB)^T = A^T B^T$.
(e) All the above statements are false.

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13. Which of the following matrices are elementary matrices?

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(a) Only B and C.
(b) Only B and D.
(c) Only B, C and D.
(d) Only D.
(e) None of the above.

14. Which of the following statements is always true?

(a) The sum of two $n \times n$ elementary matrices is an elementary matrix.
(b) The scalar product of two $n \times n$ elementary matrices is an elementary matrix.
(c) The product of two $n \times n$ elementary matrices is an elementary matrix.
(d) The inverse of an elementary matrix is an elementary matrix.
(e) None of the above.

15. Given a square matrix A, which of the following statements is **not** equivalent to A being invertible?

(a) The reduced row-echelon form of A is I.
(b) The system of linear equations $AX = B$ has a unique solution for any B.
(c) The system of linear equations $AX = 0$ has a unique solution.
(d) A is a product of elementary matrices.
(e) All the above statements are equivalent to A being invertible.

Part B: Provide your full solution on these pages. **Exam version: 0**

Question 1 (5 points) With full explanation, find (if possible) conditions on a such that the system has no solution, one solution, or infinitely many solutions.

$$\begin{array}{rcrcrcl} x & - & 2y & - & az & = & 2 \\ -ax & + & 2ay & + & 4z & = & 4 \\ x & - & y & + & az & = & 2 \end{array}$$

Write a clear and complete solution, and provide your final answer in the box below:

$$\begin{bmatrix} 1 & -2 & -a & 2 \\ -a & 2a & 4 & 4 \\ 1 & -1 & a & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & -a & 2 \\ 0 & 0 & 4-a^2 & 4+2a \\ 0 & 1 & 2a & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & a & 2 \\ 0 & 1 & 2a & 0 \\ 0 & 0 & 4-a^2 & 4+2a \end{bmatrix}$$

$$4-a^2=0 \iff a^2=4 \iff a=\pm 2$$

If $a=2$ we get $\begin{bmatrix} 1 & -2 & 2 & 2 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix}$ NO SoL.

If $a=-2$ we get $\begin{bmatrix} 1 & -2 & -2 & 2 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ Infinitely many sol.

If $a \neq \pm 2$ $(4-a^2 \neq 0)$ $\left[\begin{array}{ccc|c} 1 & -2 & +a & 2 \\ 0 & 1 & 2a & 0 \\ 0 & 0 & 1 & 4+2a/4-a^2 \end{array}\right]$ Unique sol.

ANSWER:

No solution: $a=2$

One solution: $a \neq \pm 2$

Infinitely many solutions: $a=-2$

Exam version: 0

Question 2 (5 points) Consider the matrix $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 3 & 4 \\ 1 & 0 & 2 \end{bmatrix}$.

[4] a) Use the matrix inversion algorithm to find the inverse A^{-1}.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 2 & 3 & 4 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{cccccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 3 & -2 & -2 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{cccccc} 1 & 0 & 0 & -2 & 0 & 3 \\ 0 & 3 & 0 & 0 & 1 & -2 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{cccccc} 1 & 0 & 0 & -2 & 0 & 3 \\ 0 & 1 & 0 & 0 & 1/3 & -2/3 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right]$$

$$\mathbf{A}^{-1} = \begin{bmatrix} -2 & 0 & 3 \\ 0 & 1/3 & -2/3 \\ 1 & 0 & -1 \end{bmatrix}$$

[1] b) With justification, find all solutions to the system of linear equations $AX = 0$.

$$AX = 0$$

$$A^{-1}AX = A^{-1}0 \quad \text{Since } A \text{ is invertible.}$$

$$IX = 0$$

$$X = 0 \quad \text{is the only solution}$$

End of Examination

Part B: Provide your full solution on these pages. **Exam version: 0**

Question 1 (5 points) With full explanation, find (if possible) conditions on a such that the system has no solution, one solution, or infinitely many solutions.

$$\begin{array}{rcrcrcr} x & + & 3y & + & az & = & -2 \\ ax & + & 3ay & + & 9z & = & 6 \\ x & + & 4y & - & az & = & 0 \end{array}$$

Write a clear and complete solution, and provide your final answer in the box below:

$$\begin{bmatrix} 1 & 3 & a & -2 \\ a & 3a & 9 & 6 \\ 1 & 4 & -a & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & a & -2 \\ 0 & 0 & (9-a^2) & (6+2a) \\ 0 & 1 & -2a & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & a & -2 \\ 0 & 1 & -2a & 2 \\ 0 & 0 & (9-a^2) & (6+2a) \end{bmatrix}$$

$$9-a^2=0 \iff a^2=9 \iff a=\pm 3$$

$\underline{a=+3}$ $\begin{bmatrix} 1 & 3 & 3 & -2 \\ 0 & 1 & -6 & 2 \\ 0 & 0 & 0 & 12 \end{bmatrix} \rightarrow$ No Sol

$\underline{a=-3}$ $\begin{bmatrix} 1 & 3 & -3 & -2 \\ 0 & 1 & 6 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow$ Infinitely many sol.

$a \neq \pm 3$ $(9-a^2 \neq 0)$ $\left[\begin{array}{ccc|c} 1 & 3 & a & -2 \\ 0 & 1 & -2a & 2 \\ 0 & 0 & 1 & 6+2a/9-a^2 \end{array}\right]$ Unique sol.

ANSWER:	No solution:	$a = +3$
	One solution:	$a \neq \pm 3$
	Infinitely many solutions:	$a = -3$

Exam version: 0

Question 2 (5 points) Consider the matrix $A = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 1 & 3 \\ 1 & 2 & 0 \end{bmatrix}$.

[4] a) Use the matrix inversion algorithm to find the inverse A^{-1}.

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & 0 \\ 2 & 1 & 3 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & -5 & 3 & -2 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{cccccc} 1 & 0 & 0 & -2 & 0 & 3 \\ 0 & 0 & 3 & 3 & 1 & -5 \\ 0 & -1 & 0 & -1 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 0 & 3 \\ 0 & 1 & 0 & +1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1/3 & -5/3 \end{array}\right]$$

$$\mathbf{A}^{-1} = \begin{bmatrix} -2 & 0 & 3 \\ 1 & 0 & -1 \\ 1 & 1/3 & -5/3 \end{bmatrix}$$

[1] b) With justification, find all solutions to the system of linear equations $AX = 0$.

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$AX = 0$

$A^{-1}AX = A^{-1}0$ since A is invertible.

$IX = 0$

$X = 0$ is unique sol.

End of Examination