University of Ottawa
MAT1330A Midterm Exam

November 16, 2016. Duration: 80 minutes. Instructor: Frithjof Lutscher.

Family Name: Solutions

First Name:

DGD 1 DGD 2 DGD 3 DGD 4

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- Questions 3 and 6 are short answer questions. All other questions require complete legible solutions. You need to convince me that you know why your answer is correct.
- Where it is possible to check your work, do so.
- Please do not detach the pages.
- Good luck!

Student number:____________________, Total marks: __________ out of 30

Question	1	2	3	4	5	6
Marks						

Question 1. [2 points] The function

$$f(x) = \begin{cases} \frac{x^2-1}{2x^2-x-1}, & x \neq 1, \\ k, & x = 1, \end{cases}$$

is continuous for $k = \boxed{}$.

$$\frac{x^2-1}{2x^2-x-1} = \frac{(x-1)(x+1)}{(x-1)(2x+1)}$$

Continuity: $\lim_{x \to 1} f(x) = f(1)$

$$\lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{x^2-1}{2x^2-x-1} = \lim_{x \to 1} \frac{x+1}{2x+1} = \frac{2}{3}$$

1 point for calculating the limit, 0.5 for explaining how the limit is related to continuity, and 0.5 for the correct answer

$f(1) = k$

$\Rightarrow$ Require $k = \frac{2}{3}$

Question 2. [3 points] When a disease appears in a population, health authorities record the number of infected people. One function that describes this quantity is $y(t) = 2t^3e^{-t/4}$, where $t \geq 0$ is the time in days and $y(t)$ is the number (in units of thousands of people) of infected people.

1 point for differentiating the function and finding the critical point, 1 point for explaining that it is a global max, 1 point for the value at the max, including the 1000

The infection peaks on day $\boxed{12}$

At the peak of the infection, there are $\boxed{2\cdot 12^3 e^{-3} \text{ thousand}}$ people affected.

$$y'(t) = 2\left(3t^2e^{-t/4} + t^3e^{-t/4}\cdot\left(-\tfrac{1}{4}\right)\right) = 2t^2e^{-t/4}\left(3 - \frac{t}{4}\right)$$

$y'(t) = 0$ if $t = 0$ or $t = 12$

$$\left.\begin{array}{ll} y'(t) > 0 & \text{if } 0 < t < 12 \\ y'(t) < 0 & \text{if } t > 12 \end{array}\right\} \Rightarrow \text{global max at } t = 12$$

Question 3. [2 points] Suppose that f is a function defined for all real numbers. Suppose also that f has continuous first and second derivatives for all real numbers. For each of the cases below, indicate whether the statement is true.

1/2 point per correct answer

The function f necessarily has a global minimum if...

A) ...there exists a c such that $f'(c) = 0$ and $f''(c) > 0$. true/false

B) ...$f''(x) > 0$ for all x. true/false

C) ...there exists a c such that $f'(c) = 0$ and f is concave up everywhere. true/false

D) ...there exists a c such that $f'(c) = 0$ and f' changes sign at c true/false

Question 4. [9 points] Find the derivative $y'(x)$ of the function $y(x)$. Do not simplify your answer.

(a) $y(x) = \frac{\ln(x^2)+e^{x^4}}{\sqrt{x}}$

3 points per derivative

$y'(x) = \boxed{}$

$$y'(x) = \frac{1}{|x|}\left(\left(\frac{2}{x} + 4x^3 e^{x^4}\right)\sqrt{x} + \frac{1}{2\sqrt{x}}\left(\ln(x^2) + e^{x^4}\right)\right)$$

(b) $y(x) = x^{\cos(x)}$

$y'(x) = \boxed{}$

$$y'(x) = \frac{d}{dx}\left(e^{\ln(x)\cos(x)}\right) = e^{\ln(x)\cos(x)}\left(\frac{\cos(x)}{x} - \ln(x)\sin(x)\right)$$

(c) $x^2y^2 = e^{2x+4y}$

larger!

$y'(x) = \boxed{}$

implicit: $2xy^2 + 2x^2yy' = e^{2x+4y}(2+4y')$

Solve for y': $(2x^2y - 4e^{2x+4y})y' = 2e^{2x+4y} - 2xy^2$

$$y' = \frac{2e^{2x+4y} - 2xy^2}{2x^2y - 4e^{2x+4y}}$$

Question 5. [8 points] Find the limits, using the rules from class. A table of values/the use of your calculator will not give you any points.

(a) $\lim_{x\to 0} \frac{\sqrt{1+5x}-\sqrt{1+x^2}}{x} = \boxed{}$

2 point per limit

$$= \lim_{x\to 0} \frac{\sqrt{1+5x}-\sqrt{1+x^2}}{x} \frac{\sqrt{1+5x}+\sqrt{1+x^2}}{\sqrt{1+5x}+\sqrt{1+x^2}}$$

$$= \lim_{x\to 0} \frac{1+5x-(1+x^2)}{x\left(\sqrt{1+5x}+\sqrt{1+x^2}\right)} = \lim_{x\to 0} \frac{5-x}{\sqrt{1+5x}+\sqrt{1+x^2}} = \frac{5}{2}$$

(b) $\lim_{x\to\infty} (\ln(2x^2-2) - 2\ln(x+3)) = \boxed{}$

$$= \lim_{x\to\infty} \ln\left(\frac{2x^2-2}{(x+3)^2}\right) = \ln\left(\lim_{x\to\infty} \frac{2x^2-2}{x^2+6x+9}\right)$$

$$= \ln\left(\lim_{x\to\infty} \frac{2-2/x^2}{1+6/x+9/x^2}\right) = \ln(2)$$

(c) $\lim_{x\to 0^+} \dfrac{\sqrt{3x}}{\sin(\sqrt{x})} = \boxed{}$

as $x \to 0^+$: $\sqrt{3x} \to 0$ and $\sin(\sqrt{x}) \to 0$

$\Rightarrow$ indeterminate form. $\Rightarrow$ Apply L'Hopital

$$\lim_{x\to 0^+} \frac{\sqrt{3x}}{\sin(\sqrt{x})} = \lim_{x\to 0^+} \frac{\frac{3}{2\sqrt{3x}}}{\cos(\sqrt{x})\frac{1}{2\sqrt{x}}} = \lim_{x\to 0^+} \frac{3\cdot 2\cdot\sqrt{x}}{2\cdot\sqrt{3x}\cdot\cos(\sqrt{x})}$$

$$= \lim_{x\to 0^+} \frac{3}{\sqrt{3}\cdot\cos(\sqrt{x})} = \frac{3}{\sqrt{3}} = \sqrt{3}$$

(d) $\lim_{x\to 0} \dfrac{\cos(x)-1}{x^2} = \boxed{}$

As $x \to 0$: $\cos(x) - 1 \to 0$ and $x^2 \to 0$

$\Rightarrow$ indeterminate form $\Rightarrow$ Apply L'Hopital

$$\lim_{x\to 0} \frac{\cos(x)-1}{x^2} \underset{\text{L'H}}{=} \lim_{x\to 0} \underbrace{\frac{-\sin(x)}{2x}}_{\text{indeterminate again}} \underset{\text{L'H}}{=} \lim_{x\to 0} \frac{-\cos(x)}{2} = -\frac{1}{2}$$

Question 6. [6 points] Assume that a function f as well as its first and second derivatives are continuous for all $x \in (-\infty, \infty)$. Assume that the function has the following properties:

- $f'(x) > 0$ if $|x| > 1$ and $f'(x) < 0$ if $|x| < 1$.
- $f''(x) < 0$ if $x < 0$ or $x > 2$ and $f''(x) > 0$ if $0 < x < 2$
- $\lim_{x \to \infty} f(x) = 0$ and $\lim_{x \to -\infty} f(x) = -\infty$
- $f(0) = 0$

up to 1 point per box and up to 2 for the sketch of the curve. (2 points for correct, 1 point for mostly correct and consistent with their boxed answers, 0 point for no curve or too many inconsistencies with their boxed answers)

Answer the following questions:
[You may want to make a table to summarize the properties of the function.]

(i) The graph of $f(x)$ increasing for $\boxed{|x| > 1}$

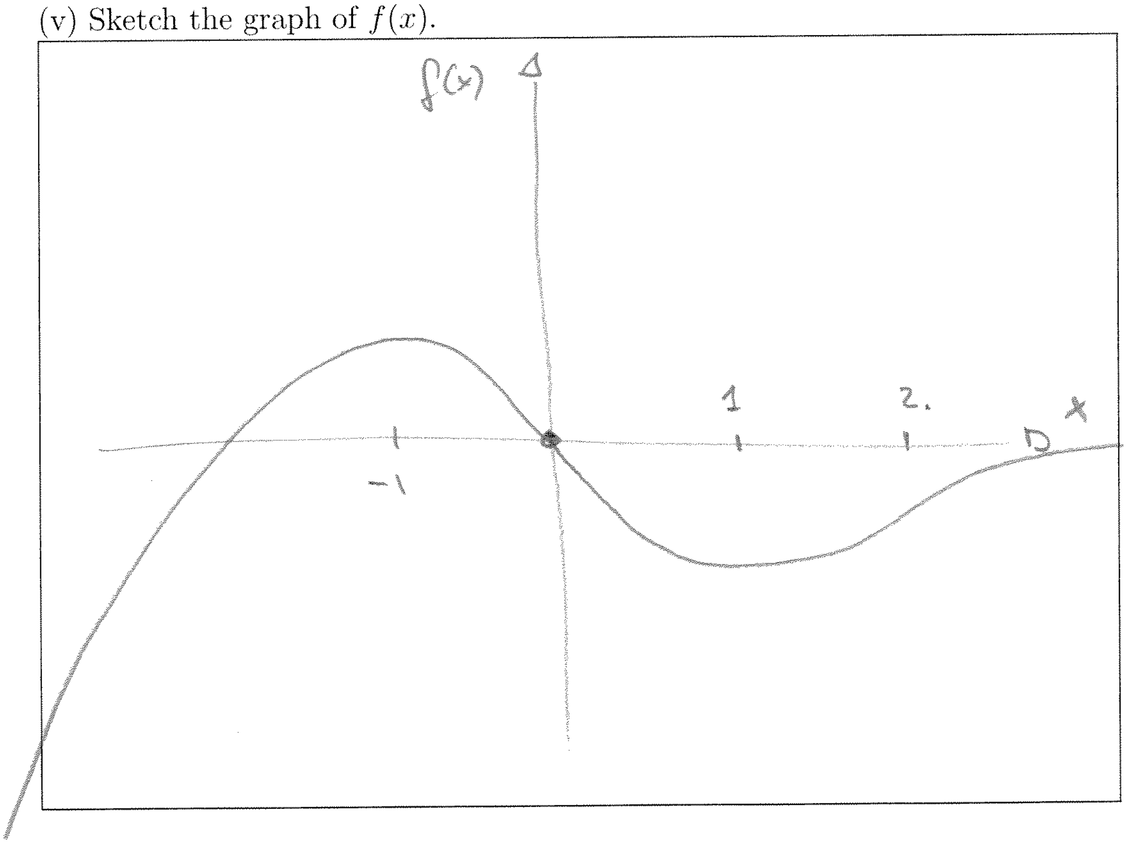
(ii) The graph of $f(x)$ concave up for $\boxed{0 < x < 2}$

re word

(iii) The graph of $f(x)$ attains a global maximum at $\boxed{-1}$

(iv) The graph of $f(x)$ attains a global minimum at $\boxed{\text{none}}$

(v) Sketch the graph of $f(x)$.



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is continuous for $k = \boxed{5/3}$.

$$\frac{2x^2-3x-2}{x^2-x-2} = \frac{(x-2)(2x+1)}{(x-2)(x+1)}$$

Question 2. [3 points] When a disease appears in a population, health authorities record the number of infected people. One function that describes this quantity is $y(t) = 3t^4 e^{-t/4}$, where $t \geq 0$ is the time in days and $y(t)$ is the number (in units of thousands of people) of infected people.

The infection peaks on day $\boxed{16}$

At the peak of the infection, there are $\boxed{3 \cdot 16^4 \cdot e^{-4} \text{ thousand}}$ people affected.

$$y'(t) = 3t^3 e^{-t/4}\left(4 - \frac{t}{4}\right)$$

Question 3. [2 points] Suppose that f is a function defined for all real numbers. Suppose also that f has continuous first and second derivatives for all real numbers. For each of the cases below, indicate whether the statement is true.

The function f necessarily has a global minimum if...

A) ...$f''(x) > 0$ for all x. true/false

B) ...there exists a c such that $f'(c) = 0$ and f is concave up everywhere. true/false

C) ...there exists a c such that $f'(c) = 0$ and f' changes sign. at c true/false

D) ...there exists a c such that $f'(c) = 0$ and $f''(c) > 0$. true/false

Question 4. [9 points] Find the derivative $y'(x)$ of the function $y(x)$. Do not simplify your answer.

(a) $y(x) = \frac{\ln(x^3)+e^{x^2}}{\sqrt{x}}$

$y'(x) =$ $\boxed{}$

$$y'(x) = \frac{1}{|x|}\left(\left(\frac{3}{x} + 2xe^{x^2}\right)\sqrt{x} + \frac{1}{2\sqrt{x}}\left(\ln(x^3) + e^{x^2}\right)\right)$$

(b) $y(x) = x^{\tan(x)}$

$y'(x) =$ $\boxed{}$

$$y'(x) = \frac{d}{dx}\left(e^{\ln(x)\tan(x)}\right) = e^{\ln(x)\tan(x)}\left(\frac{\tan(x)}{x} + \ln(x)\sec^2(x)\right)$$

(c) $x^3y^2 = e^{3x+2y}$

$y'(x) =$ $\boxed{}$

$$3x^2y^2 + 2x^3yy' = e^{3x+2y}(3+2y')$$

$$y' = \frac{3e^{3x+2y} - 3x^2y^2}{2x^3y - 2e^{3x+2y}}$$

Question 5. [8 points] Find the limits, using the rules from class. A table of values/the use of your calculator will not give you any points.

(a) $\lim_{x\to 0} \frac{\sqrt{1+3x}-\sqrt{1+x^2}}{x} = \boxed{3/2}$

$$\frac{(\sqrt{1+3x}-\sqrt{1+x^2})(\sqrt{1+3x}+\sqrt{1+x^2})}{x\,(\sqrt{1+3x}+\sqrt{1+x^2})}$$

$$= \frac{3-x}{\sqrt{1+3x}+\sqrt{1+x^2}}$$

(b) $\lim_{x\to\infty} (\ln(4x^2-2) - 2\ln(x+1)) = \boxed{\ln(4)}$

$$\ln \frac{4x^2-2}{x^2+2x+1}$$

(c) $\lim_{x \to 0^+} \frac{\sqrt{5x}}{\sin(\sqrt{x})} = \boxed{}$

$$\lim_{x \to 0^+} \frac{\sqrt{5x}}{\sin(\sqrt{x})} \underset{\text{L'H}}{=} \lim_{x \to 0^+} \frac{\frac{5}{2\sqrt{5x}}}{\cos(\sqrt{x})\,\frac{1}{2\sqrt{x}}} = \lim_{x \to 0^+} \frac{5 \cdot 2\sqrt{x}}{2\sqrt{5x}\,\cos(\sqrt{x})} = \lim_{x \to 0^+} \frac{5}{\sqrt{5}\cos(\sqrt{x})}$$

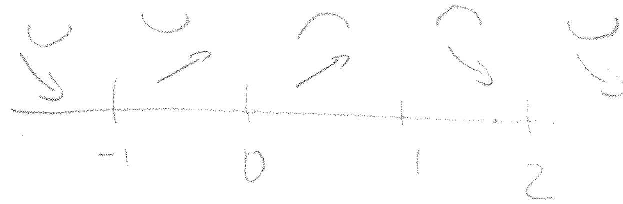
$$= \sqrt{5}$$

(d) $\lim_{x \to 0} \frac{x^2}{\cos(x) - 1} = \boxed{}$

$$\lim_{x \to 0} \frac{x^2}{\cos(x)-1} \underset{\text{L'H}}{=} \lim_{x \to 0} \frac{2x}{-\sin(x)} \underset{\text{L'H}}{=} \lim_{x \to 0} \frac{2}{-\cos(x)} = -2$$

Question 6. [6 points] Assume that a function f as well as its first and second derivatives are continuous for all $x \in (-\infty, \infty)$. Assume that the function has the following properties:

- $f'(x) > 0$ if $|x| < 1$ and $f'(x) < 0$ if $|x| > 1$.
- $f''(x) > 0$ if $x < 0$ or $x > 2$ and $f''(x) < 0$ if $0 < x < 2$
- $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to-\infty} f(x) = \infty$
- $f(0) = 0$



Answer the following questions:
[You may want to make a table to summarize the properties of the function.]

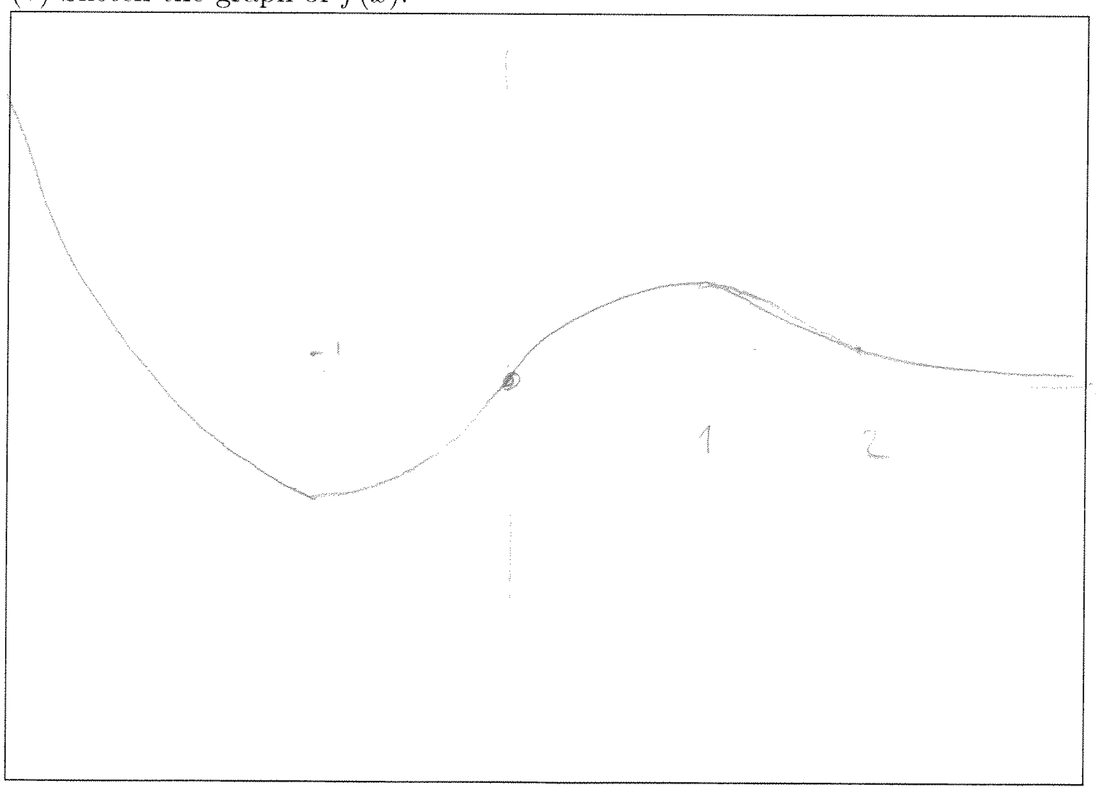
(i) The graph of $f(x)$ increasing for $\boxed{|x| < 1}$

(ii) The graph of $f(x)$ concave up for $\boxed{x < 0 \text{ or } x > 2}$

(iii) The graph of $f(x)$ attains a global maximum at $\boxed{\text{not at all}}$

(iv) The graph of $f(x)$ attains a global minimum at $\boxed{x = -1}$

(v) Sketch the graph of $f(x)$.



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is continuous for $k = \boxed{3/5}$.

$$\frac{x^2-x-2}{2x^2-3x-2} = \frac{(x-2)(x+1)}{(x-2)(2x+1)}$$

Question 2. [3 points] When a disease appears in a population, health authorities record the number of infected people. One function that describes this quantity is $y(t) = 5t^4e^{-t/3}$, where $t \geq 0$ is the time in days and $y(t)$ is the number (in units of thousands of people) of infected people.

The infection peaks on day $\boxed{}$

At the peak of the infection, there are $\boxed{}$ people affected.

$5t^3e^{-t/3}\left(4 - \frac{t}{3}\right)$ $\quad$ $t = 12$

Question 3. [2 points] Suppose that f is a function defined for all real numbers. Suppose also that f has continuous first and second derivatives for all real numbers. For each of the cases below, indicate whether the statement is true.

The function f necessarily has a global minimum if...

$A)$ ...there exists a c such that $f'(c) = 0$ and f is concave up everywhere. true/false

$B)$ ...there exists a c such that $f'(c) = 0$ and f' changes sign at c true/false

$C)$ ...there exists a c such that $f'(c) = 0$ and $f''(c) > 0$. true/false

$D)$ ...$f''(x) > 0$ for all x. true/false

Question 4. [9 points] Find the derivative $y'(x)$ of the function $y(x)$. Do not simplify your answer.

(a) $y(x) = \frac{\ln(x^2)+e^{x^3}}{\sqrt{x}}$

$y'(x) = \boxed{}$

$$y'(x) = \frac{1}{|x|}\left[\left(\frac{2}{x} + 3x^2e^{x^3}\right)\sqrt{x} + \frac{1}{2\sqrt{x}}\left(\ln(x^2) + e^{x^3}\right)\right]$$

(b) $y(x) = x^{\sin(x)}$

$y'(x) = \boxed{}$

$$\frac{d}{dx} e^{\ln(x)\sin(x)} = e^{\ln(x)\sin(x)}\left(\frac{\sin(x)}{x} + \ln(x)\cos(x)\right)$$

(c) $x^2y^3 = e^{x+3y}$

$y'(x) = \boxed{}$

$$2xy^3 + 3x^2y^2y' = e^{x+3y}(1+3y')$$

$$y' = \frac{e^{x+3y} - 2xy^3}{3x^2y^2 - 3e^{x+3y}}$$

Question 5. [8 points] Find the limits, using the rules from class. A table of values/the use of your calculator will not give you any points.

(a) $\lim_{x \to 0} \frac{\sqrt{1+4x} - \sqrt{1+x^2}}{x} = \boxed{2}$

$$\frac{\sqrt{1+4x} - \sqrt{1+x^2}}{x} \cdot \frac{\sqrt{1+4x} + \sqrt{1+x^2}}{\sqrt{1+4x} + \sqrt{1+x^2}}$$

$$= \frac{4x - x^2}{x\left(\sqrt{1+4x} + \sqrt{1+x^2}\right)} = \frac{4 - x}{\sqrt{1+4x} + \sqrt{1+x^2}} \longrightarrow \frac{4}{2} = 2$$

(b) $\lim_{x \to \infty} (\ln(5x^2+3) - 2\ln(x-1)) = \boxed{\ln(5)}$

$$\ln \frac{5x^2+3}{x^2-2x+1} = \ln \frac{5 + 3/x^2}{1 - 2/x + 1/x^2}$$

(c) $\displaystyle\lim_{x\to 0^+} \frac{\sqrt{6x}}{\sin(\sqrt{x})} = \boxed{}$

$$\lim_{x\to 0^+} \frac{\sqrt{6x}}{\sin(\sqrt{x})} \underset{\substack{\uparrow \\ L'H}}{=} \lim_{x\to 0^+} \frac{\frac{6}{2\sqrt{6x}}}{\cos(\sqrt{x})\,\frac{1}{2\sqrt{x}}} = \lim_{x\to 0^+} \frac{6\cdot 2\sqrt{x}}{2\sqrt{6x}\,\cos(\sqrt{x})}$$

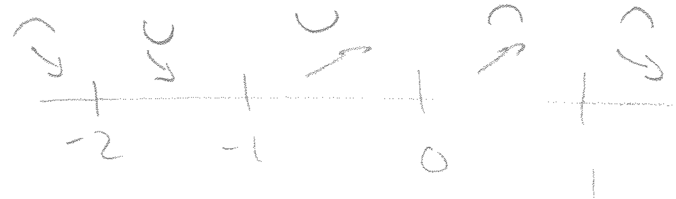
$$= \lim_{x\to 0^+} \frac{6}{\sqrt{6}\cos(\sqrt{x})} = \sqrt{6}$$

(d) $\displaystyle\lim_{x\to 0} \frac{x^2}{1-\cos(x)} = \boxed{}$

$$\lim_{x\to 0} \frac{x^2}{1-\cos(x)} \underset{\substack{\uparrow \\ L'H}}{=} \lim_{x\to 0} \frac{2x}{\sin(x)} \underset{\substack{\uparrow \\ L'H}}{=} \lim_{x\to 0} \frac{2}{\cos(x)} = 2$$

Question 6. [6 points] Assume that a function f as well as its first and second derivatives are continuous for all $x \in (-\infty, \infty)$. Assume that the function has the following properties:

- $f'(x) > 0$ if $|x| < 1$ and $f'(x) < 0$ if $|x| > 1$.
- $f''(x) < 0$ if $x < -2$ or $x > 0$ and $f''(x) > 0$ if $-2 < x < 0$
- $\lim_{x\to\infty} f(x) = -\infty$ and $\lim_{x\to-\infty} f(x) = 0$
- $f(0) = 0$



Answer the following questions:
[You may want to make a table to summarize the properties of the function.]

(i) The graph of $f(x)$ increasing for $\boxed{|x| < 1}$

(ii) The graph of $f(x)$ concave up for $\boxed{-2 < x < 0}$

(iii) The graph of $f(x)$ attains a global maximum at $\boxed{x = 1}$

(iv) The graph of $f(x)$ attains a global minimum at $\boxed{\text{not at all}}$

(v) Sketch the graph of $f(x)$.