University of Ottawa
MAT1330A Midterm Exam

October 5, 2016. Duration: 80 minutes. Instructor: Frithjof Lutscher.

Family Name: Solutions

First Name:____________________

DGD 1 DGD 2 DGD 3 DGD 4

Do **not** write your student ID number on this front page. Please write your student ID number in the space provided on the second page.

Take your time to read the entire paper before you begin to write, and read each question carefully. Remember that certain questions are worth more points than others. Make a note of the questions that you feel confident you can do, and then do those first: you do not have to proceed through the paper in the order given.

- You have 80 minutes to complete this exam.
- This is a closed book exam, and no notes of any kind are allowed. The use of cell phones, pagers or any text storage or communication device **is not permitted.**
- Only the Faculty approved calculators (TI-30X, TI-34X, Casio FX-260X and Casio FX-300X) are allowed.
- Questions 1–10 are multiple choice questions. You MUST record your answers in the boxes at the top of page 2. Each of these questions is worth 1 point.
- Questions 11 and 12 are long-answer questions. The correct answer requires justification written legibly and logically: you must convince me that you know why your solution is correct. Answer these questions in the space provided. Use the backs of pages for additional calculations if necessary.
- Where it is possible to check your work, do so.
- Please do not detach the pages.
- Good luck!

Student number:____________________, Total marks: __________ out of 30

Problem	1	2	3	4	5	6	7	8	9	10	11	12
Answers to 1–10	B	B	E	E	E	C	E	D	B	C	X	X
Marks												

Question 1. If $f(x) = 4x - 2$ and $g(x) = -5x + 9$ what is $(f \circ g)(1)$?

$g(1) = 4$

$f(g(1)) = f(4) = 14$

Answer: A: -14; B: 14; C: -1; D: 1; E: 0.

Question 2. What is the domain of the function

$$f(x) = \frac{x+1}{x^2 - 3}$$

$x^2 - 3 \neq 0$

$x^2 \neq 3 \Rightarrow x \neq \pm\sqrt{3}$

Answer:
A: all numbers except -1; B: all numbers except $\pm\sqrt{3}$; C: all numbers $|x| > 3$;
D: all numbers except ± 3; E: all numbers except 0

Question 3. Which of the following statements is true?
(i) If $a, b > 0$ then $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$. ×
(ii) $\arccos(x) = \frac{1}{\cos(x)}$ ×
(iii) If $a, b > 0$ then $\ln(a) + \ln(b) = \ln(ab)$. ✓
(iv) $(a+b)^2 = a^2 + b^2$ ×
(v) $e^{x+y} = e^x + e^y$ ×

Answer: A: (i), (ii) and (iii); B: (ii) and (iv); C: (ii), (iii) and (v);
D: (ii) and (iii) only; E: (iii) only.

Question 4. Find all solutions of the equation $|x^3 - 7| = 7$.

Answer: A: 0; B: $0, \pm\sqrt[3]{14}$; C: $0, \pm\sqrt{7}$; D: $0, \pm\sqrt[3]{7}$; E: $0, \sqrt[3]{14}$

$x^3 > 7 \Rightarrow x^3 - 7 = 7$

$x^3 = 14$

$x = \sqrt[3]{14}$

$x^3 < 7 \Rightarrow -(x^3 - 7) = 7$

$x^3 = 0$

$x = 0$

Question 5. Find all solutions of $\log(x + 2) + \log(x + 3) = \log(2)$.

Answer: A: -4; B: $1, -4$; C: $1, 4$; D: $-1, -4$; E: -1

$(x+2)(x+3) = 2$

$x^2 + 5x + 4 = 0$

$(x+4)(x+1) = 0$

$x = -4$

$x = -1$

but $x = -4$ does not work

Question 6. Determine the period and the amplitude of the function $f(t) = 5\pi \sin(3t)$.

Answer:

A: amplitude 5π, period $\pi/3$.
B: amplitude 5, period $\pi/3$.
C: amplitude 5π, period $2\pi/3$.
D: amplitude 5, period $2\pi/3$.
E: amplitude 5π, period $\pi/6$.

Amplitude

Period: $3t = 2\pi$

$t = \frac{2\pi}{3}$

Question 7. If x and y are non-zero numbers, then $\frac{5}{x} - \frac{6}{y}$ equals

Answer: A: $\frac{5x+6y}{xy}$; B: $\frac{-30}{xy}$; C: $\frac{-1}{x-y}$; D: $\frac{6x-5y}{xy}$; E: $\frac{5y-6x}{xy}$;

$\frac{5y}{xy} - \frac{6x}{xy} = \frac{5y-6x}{xy}$

Question 8. The quadratic $2x^2 + x - 6$ factors as

Answer: A: $(2x+3)(x+2)$; B:$(x+3)(x-2)$; C: $(2x-2)(x+3)$; D: $(2x-3)(x+2)$;
E: $(2x+3)(x-2)$

$= 2x^2 \underbrace{- 3x + 4x}_{+x} - 6$

Question 9. What is the range of the function $f(x) = -4 + 2\sin(6x - 5)$?

Answer: A: $[-4, -2]$; B: $[-6, -2]$; C: $[-3, -5]$; D: $[-2, 2]$; E: $[-4, 2]$

$\sin(x)$ ranges in $[-1,1]$ $\quad -4 \pm 2 = \begin{cases} -6 \\ -2 \end{cases}$

Question 10. Which of the following statements is true?

(i) Every DTDS has at least one steady state. ✗ for example $x_{t+1} = x_t + 1$
(ii) Every DTDS has at least one stable steady state. ✗ see (i)
(iii) A DTDS can have infinitely many steady states. ✓ $x_{t+1} = x_t$
(iv) No DTDS can have infinitely many stable steady states. ✗ $x_{t+1} = \sin(x_t) + x_t$
(v) There is a DTDS where every point is a steady state. ✓

Answer: A: (i) and (iv); B: (i), (ii) and (iv); C: (iii) and (v);
D: (i), (iv) and (v); E: (iii) only.

Question 11. [14 points] Mono Lake in California is a rare example of a lake that has no outflow of water. Tributaries provide the lake with 150 m^3 of water per day. Due to the intense sun, 2% of the water in the lake evaporates per day. Denote by V_t the volume (in m^3) in the lake on day t. The DTDS for the volume is given by $V_{t+1} = 0.98V_t + 150$.

(a) The updating function of the DTDS is $f(x) = \boxed{0.98x + 150}$

(b) The steady state of the DTDS is $V^* = \boxed{\frac{150}{0.02} = 7500}$

(c) The general solution of the DTDS is $V_t = \boxed{(0.98)^t (V_0 - V^*) + V^*}$

(d) One year, an expert team estimates the volume of the lake at $V_0 = 5000$ m^3. Calculate the expected volume for the two subsequent days.

Answer $\boxed{V_1 = 5050, \quad V_2 = 5099}$

(e) How many days after this estimate would it take for the lake volume to grow to 80% of the steady state value?

Answer $\boxed{\text{26 days}}$

Solution: $V_t = (0.98)^t (V_0 - V^*) + V^*$

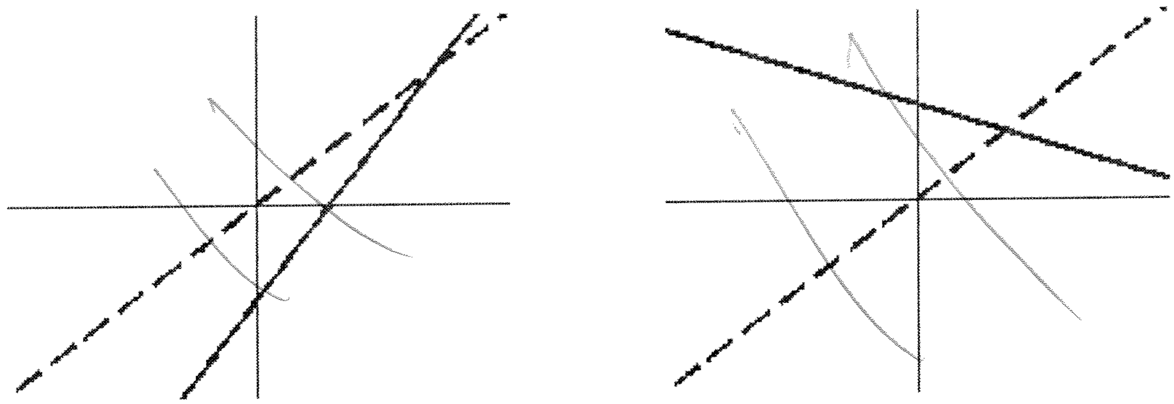
Want 80% of steady state : $V_t = 0.8 \cdot V^*$

$\Rightarrow$ Solve $0.8V^* = (0.98)^t (V_0 - V^*) + V^*$ for t

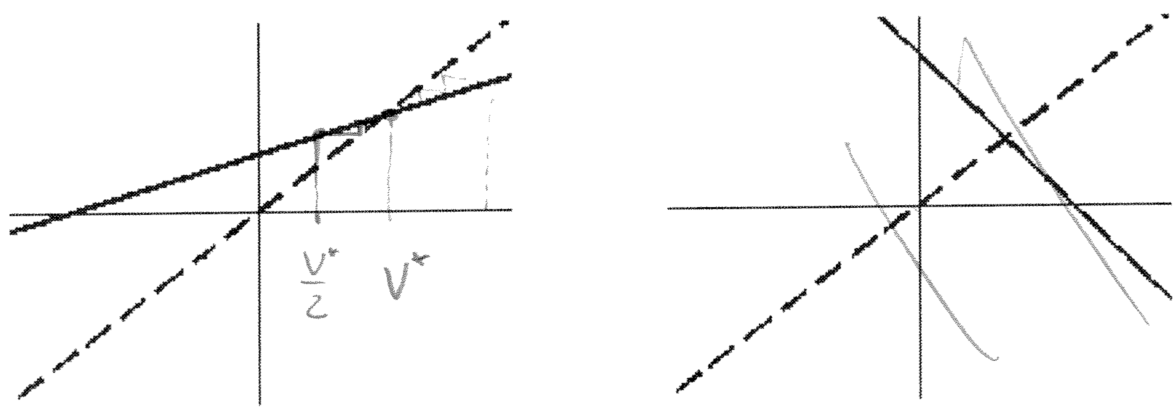
$$\Rightarrow t = \frac{\ln\left(\frac{-0.2V^*}{V_0 - V^*}\right)}{\ln(0.98)} = 25.285$$

Use $V_0 = 5000$ from d)
$V^* = 7500$ from b)

(f) In each of the four plots below the coordinate axes are plotted as thin solid lines; the diagonal is plotted as a thin dashed line and the graph of a function is plotted as a thick line. The updating function of the DTDS for V_t corresponds to exactly one of these plots. In it, indicate the value of the steady state and draw two iterations of a cobweb, starting at half of the steady state value. Cross out the other three plots.



Since $f(x) = 0.98x + 150$:



(g) Is the steady state stable? Justify your answer by **two different** ways of reasoning.

Yes: 1) the iterations in the cobweb approach V^*

2) the DTDS is linear and $|r| = |0.98| < 1$

$\Rightarrow$ the theorem from class says that V^* is stable.

(h) Due to global warming, daily evaporation increases to 4%. At the same time, the city of Los Angeles is in dire need of water and wants to extract water from Mono lake for its needs. How much can the city take per day if the total volume must remain above 3000 m^3 in the long run (i.e. the steady state value needs to be at least 3000 m^3).

Answer: The city can extract no more than $\boxed{30}$ m^3 per day.

1) Find required input for $V^* = 3000$ and $r = 1 - 0.04 = 0.96$

Since $V^* = \frac{d}{1-r}$ we have $d = (1-r)V^* = 0.04 \cdot 3000 = 120.$

2) Input from tributaries (streams and rivers) is $150\ m^3/\text{day}$

3) The city can take the difference $150 - 120 = 30\ m^3/\text{day}$

Question 12. [6 points] Consider the DTDS

$$x_{t+1} = r x_t \left(1 - \frac{x_t}{K}\right)^2$$

where $r, K > 0$ are parameters.

(a) Calculate the steady state values for this DTDS. [Hint: some of them may depend on parameter values.]

Answer $\boxed{x = 0, \quad x = K\left(1 \pm \sqrt{1/r}\right)}$

Solve $x = rx\left(1 - \frac{x}{K}\right)^2$. $x = 0$ is a solution.

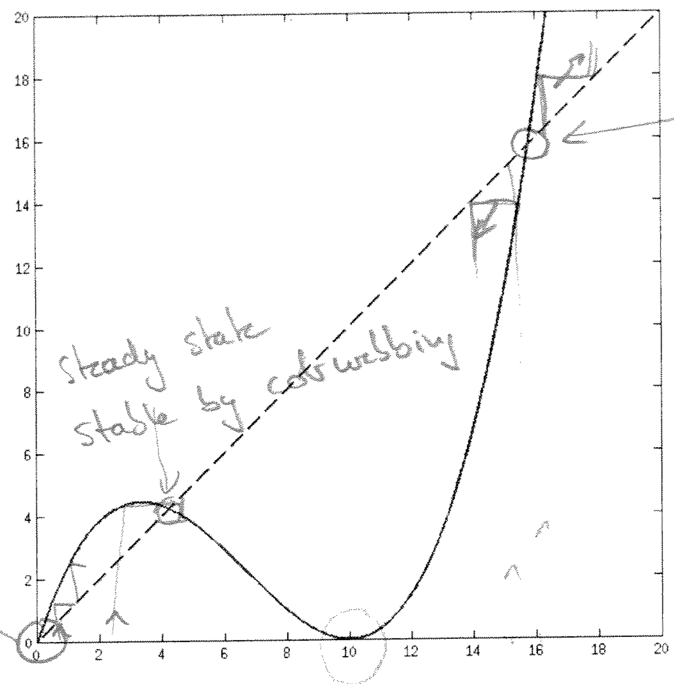
The others: $\left(1 - \frac{x}{K}\right)^2 = \frac{1}{r} \Rightarrow x = K\left(1 \pm \sqrt{\frac{1}{r}}\right)$

quadratic formula

(b) The plot below shows the updating function and the line $x = y$ for parameter $r = 3$. What is the value of K used in the plot?

Answer: $K = \boxed{10}$ $f(x) = rx\left(1 - \frac{x}{K}\right)^2$ has zeros at 0 and K.

(c) In the plot, indicate all the steady states and determine their stability.



steady state. unstable by cobwebbing

unstable by cobwebbing

(d) Give the expression for the updating function of the two-time step DTDS $x_{t+2} = g(x_t)$. [Do not simplify your answer.]

Answer: $g(x) = \boxed{r^2 x_t\left(1 - \frac{x_t}{K}\right)^2\left(1 - \frac{r}{K}x_t\left(1 - \frac{x_t}{K}\right)^2\right)^2}$

$$g(x_{t+2}) = r x_{t+1}\left(1 - \frac{x_{t+1}}{K}\right)^2$$

$$= r^2\left(x_t\left(1 - \frac{x_t}{K}\right)^2\right)\left(1 - \frac{1}{K}rx_t\left(1 - \frac{x_t}{K}\right)^2\right)^2$$

Student number:____________________, Total marks: __________ out of 30

Problem	1	2	3	4	5	6	7	8	9	10	11	12
Answers to 1–10	A	C	B	E	B	C	E	A	A	D	X	X
Marks												

Question 1. If $f(x) = 3x - 5$ and $g(x) = -2x + 7$ what is $(f \circ g)(1)$?

Answer: A: 10; B: -10; C: -11; D: 11; E: 0.

Question 2. What is the domain of the function

$$f(x) = \frac{x+1}{x^2 - 5}$$

Answer:
A: all numbers except -1; B: all numbers $|x| > 5$; C: all numbers except $\pm\sqrt{5}$;
D: all numbers except ± 5; E: all numbers except 0

Question 3. Which of the following statements is true?
(i) If $a, b > 0$ then $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$.
(ii) $\arccos(x) = \frac{1}{\cos(x)}$
(iii) If $a, b > 0$ then $\ln(a) + \ln(b) = \ln(a+b)$.
(iv) $(a+b)^2 = a^2 + b^2$
(v) $e^{x+y} = e^x e^y$

Answer: A: (i), (ii) and (iii); B: (v) only; C: (ii), (iii) and (v);
D: (ii) and (iii) only; E: (ii) and (iv).

Question 4. Find all solutions of the equation $|x^5 - 7| = 7$.

Answer: A: $0, \pm\sqrt[5]{14}$; B: 0; C: $0, \pm\sqrt{7}$; D: $0, \pm\sqrt[5]{7}$; E: $0, \sqrt[5]{14}$

Question 5. Find all solutions of $\log(x+4) + \log(x+3) = \log(2)$.

Answer: A: -5; B: -2; C: $2, 5$; D: $-2, -5$; E: $-2, 5$

Question 11. [14 points] Mono Lake in California is a rare example of a lake that has no outflow of water. Tributaries provide the lake with 240 m^3 of water per day. Due to the intense sun, 6% of the water in the lake evaporates per day. Denote by V_t the volume (in m^3) in the lake on day t. The DTDS for the volume is given by $V_{t+1} = 0.94V_t + 240$.

(a) The updating function of the DTDS is $f(x) = \boxed{0.94x + 240}$

(b) The steady state of the DTDS is $V^* = \boxed{\frac{240}{0.06} = 4000}$

(c) The general solution of the DTDS is $V_t = \boxed{(0.94)^t (V_0 - V^*) + V^*}$

(d) One year, an expert team estimates the volume of the lake at $V_0 = 2000$ m^3. Calculate the expected volume for the two subsequent days.

Answer $\boxed{V_1 = 2120, \quad V_2 = 2232.8}$

(e) How many days after this estimate would it take for the lake volume to grow to 80% of the steady state value?

Answer $\boxed{15}$

$t = 14.8086$

(h) Due to global warming, daily evaporation increases to 10%. At the same time, the city of Los Angeles is in dire need of water and wants to extract water from Mono lake for its needs. How much can the city take per day if the total volume must remain above 2000 m^3 in the long run (i.e. the steady state value needs to be at least 2000 m^3).

Answer: The city can extract no more than $\boxed{40}$ m^3 per day.

Question 12. [6 points] Consider the DTDS

$$z_{t+1} = r z_t \left(1 - \frac{z_t}{L}\right)^2$$

where $r, L > 0$ are parameters.

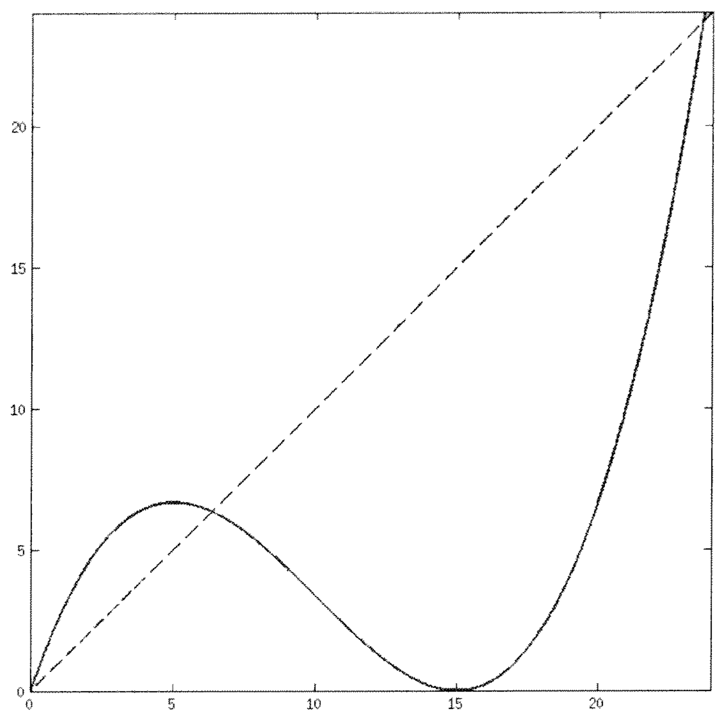
(a) Calculate the steady state values for this DTDS. [Hint: some of them may depend on parameter values.]

Answer $\boxed{}$

(b) The plot below shows the updating function and the diagonal for parameter $r = 3$. What is the value of L used in the plot?

Answer: $L =$ 15

(c) In the plot, indicate all the steady states and determine their stability.



(d) Give the expression for the updating function of the two-time step DTDS $z_{t+2} = g(z_t)$. [Do not simplify your answer.]

Answer: $g(z) =$

Student number:____________________, Total marks: __________ out of 30

Problem	1	2	3	4	5	6	7	8	9	10	11	12
Answers to 1–10	D	E	D	B	A	C	B	B	E	E	X	X
Marks												

Question 1. If $f(x) = 5x - 3$ and $g(x) = -2x + 6$ what is $(f \circ g)(1)$?

Answer: A: -17; B: -2; C: 2; D: 17; E: 0.

Question 2. What is the domain of the function

$$f(x) = \frac{x+1}{x^2 - 5}$$

Answer:
A: all numbers except 0; B: all numbers except -1; C: all numbers $|x| > 5$;
D: all numbers except ± 5; E: all numbers except $\pm\sqrt{5}$

Question 3. Which of the following statements is true?
(i) If $a, b > 0$ then $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$.
(ii) $\arccos(x) = \frac{1}{\cos(x)}$
(iii) If $a, b > 0$ then $\ln(a) + \ln(b) = \ln(ab)$.
(iv) $(a+b)^2 = a^2 + b^2$
(v) $e^{x+y} = e^x e^y$

Answer: A: (i), (ii) and (iii); B: (ii) and (iv); C: (ii), (iii) and (v);
D: (iii) and (v) only; E: (iii) only.

Question 4. Find all solutions of the equation $|x^3 - 6| = 6$.

Answer: A: 0; B: $0, \sqrt[3]{12}$; C: $0, \pm\sqrt{6}$; D: $0, \pm\sqrt[3]{6}$; E: $0, \pm\sqrt[3]{12}$

Question 5. Find all solutions of $\log(x+5) + \log(x+3) = \log(3)$.

Answer: A: -2; B: $2, -6$; C: $2, 6$; D: $-2, -6$; E: 6

Question 11. [14 points] Mono Lake in California is a rare example of a lake that has no outflow of water. Tributaries provide the lake with 200 m^3 of water per day. Due to the intense sun, 4% of the water in the lake evaporates per day. Denote by V_t the volume (in m^3) in the lake on day t. The DTDS for the volume is given by $V_{t+1} = 0.96V_t + 200$.

(a) The updating function of the DTDS is $f(x) =$ $\boxed{0.96x + 200}$

(b) The steady state of the DTDS is $V^* =$ $\boxed{\frac{200}{0.04} = 5000}$

(c) The general solution of the DTDS is $V_t =$ $\boxed{(0.96)^t (V_0 - V^*) + V^*}$

(d) One year, an expert team estimates the volume of the lake at $V_0 = 3000$ m^3. Calculate the expected volume for the two subsequent days.

Answer $\boxed{V_1 = 3080, \quad V_2 = 3156}$

(e) How many days after this estimate would it take for the lake volume to grow to 80% of the steady state value?

Answer $\boxed{17}$

$t = 16.9797$

(h) Due to global warming, daily evaporation increases to 7%. At the same time, the city of Los Angeles is in dire need of water and wants to extract water from Mono lake for its needs. How much can the city take per day if the total volume must remain above 2000 m^3 in the long run (i.e. the steady state value needs to be at least 2000 m^3).

Answer: The city can extract no more than $\boxed{60}$ m^3 per day.

Question 12. [6 points] Consider the DTDS

$$y_{t+1} = r y_t \left(1 - \frac{y_t}{S}\right)^2$$

where $r, S > 0$ are parameters.

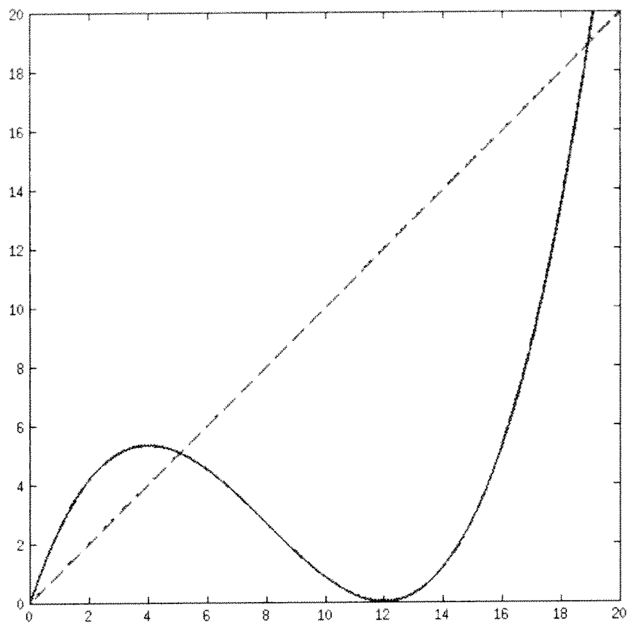
(a) Calculate the steady state values for this DTDS. [Hint: some of them may depend on parameter values.]

Answer $\boxed{}$

(b) The plot below shows the updating function and the diagonal for parameter $r = 3$. What is the value of S used in the plot?

Answer: $S =$ 12

(c) In the plot, indicate all the steady states and determine their stability.



(d) Give the expression for the updating function of the two-time step DTDS $y_{t+2} = g(y_t)$. [Do not simplify your answer.]

Answer: $g(y) =$