MAT 1330, Fall 2016 Assignment 6
Due Thursday December 1 by 8:00pm.
Late assignments will not be accepted; nor will unstapled assignments. Professors in the math department will not lend you a stapler; do not ask for one.
Please print this assignment double sided: to save paper and to ease weight.

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By signing below, we declare that this work is our own and that we have not copied from any other individual or other source.

Signatures ____

QUESTION 1. The dynamics of a fish stock in the presence of harvesting are given by the DTDS

$$x_{t+1} = 6x_t(1-2x_t) - hx_t,$$

where the first term on the right hand side describes reproduction and the second harvesting with intensity $h > 0$.

The steady states of the DTDS are $\boxed{x_1^* = 0, \quad x_2^* = \frac{5-h}{12}}$

Solve $x = 6x(1-2x) - hx$ for x.

Clearly $x_1^* = 0$ is a solution. For second solution: $1 = 6(1-2x) - h$

$\Leftrightarrow \frac{1+h}{6} = 1 - 2x \Leftrightarrow x_2^* = \frac{1}{2}\left(1 - \frac{1+h}{6}\right) = \frac{5-h}{12}$

If the harvesting intensity h is in the range $\boxed{0 < h < 5}$, then the DTDS has a positive steady state.

$x_2^* = \frac{5-h}{12}$, $\quad x_2^* > 0 \iff 5-h > 0 \iff h < 5$

The positive steady state is stable if h is in the range $\boxed{3 < h < 5}$.

$f(x) = 6x - 12x^2 - hx$

$f'(x) = 6 - 24x - h$

$f'(x_2^*) = 6 - 24\frac{5-h}{12} - h = h - 4$

$|f'(x_2^*)| < 1$

$\iff -1 < h-4 < 1$

$\iff 3 < h < 5$

The yield at steady state ($Y = hx^*$) is maximized for $h = \boxed{5/2}$.

$Y = hx_2^* = \frac{h(5-h)}{12}$ $\qquad Y'(h) = \frac{5-2h}{12}$

$\left.\begin{array}{l} Y'(h) > 0 \text{ if } h < \frac{5}{2} \\ Y'(h) < 0 \text{ if } h > \frac{5}{2} \end{array}\right\}$ Global max at $h = 5/2$

Is the steady state stable at optimal yield? Answer: $\boxed{\text{no}}$

Since $\frac{5}{2} \notin (3,5)$

QUESTION 2. Consider the function $f(x) = \sqrt[5]{x}$.

The Taylor polynomial of degree 3 at base point a is given by

$$T_3(x) = \boxed{\sqrt[5]{a} + \frac{x-a}{5a^{4/5}} - \frac{2(x-a)^2}{25\,a^{9/5}} + \frac{6(x-a)^3}{125\,a^{14/5}}}.$$

$f(x) = x^{1/5}$, $f'(x) = \frac{1}{5}x^{-4/5}$, $f''(x) = -\frac{4}{25}x^{-9/5}$, $f'''(x) = \frac{36}{125}x^{-14/5}$

The approximation of $\sqrt[5]{40}$ based on

- linear approximation is $T_1(40) = \boxed{2.1}$;
- quadratic approximation is $T_2(40) = \boxed{2.09}$;
- cubic approximation is $T_3(40) = \boxed{2.0915}$;

[Hint: choose the base point carefully.]

Choose $a = 32 = 2^5$. Then $f(a) = 2$ and $x - a = 8 = 2^3$

Also: $f'(a) = \frac{1}{5 \cdot 16}$, $f''(a) = \frac{4}{25 \cdot 512}$, $f'''(a) = \frac{36}{125 \cdot 2^{14}}$

$$T_3(x) = 2 + \frac{x-32}{5 \cdot 16} - \frac{2(x-32)^2}{25 \cdot 512} + \frac{6(x-32)^3}{125 \cdot 2^{14}}$$

$$= \underbrace{\underbrace{2 + \frac{1}{10}}_{=2.1} - \frac{1}{100}}_{=2.09} + \frac{3}{125 \cdot 16} = 2.0915$$

QUESTION 3. Consider the functions

$$f(x) = \frac{2}{1+x^2}, \qquad g(x) = e^x.$$

(a) Show that there is a value $\bar{x}$ with $0 \le \bar{x} \le 1$ and $f(\bar{x}) = g(\bar{x})$.

Answer: Consider $F(x) = f(x) - g(x)$. Need to find $\bar{x}$ s.t. $F(\bar{x}) = 0$

1) F is continuous since f and g are

2) $F(0) = 2 - 1 = 1 > 0$, $F(1) = 1 - e < 0$ $\Rightarrow$ sign change.

3) By the IVT, there is $\bar{x} \in [0,1]$ s.t. $F(\bar{x}) = 0$.

Then $f(\bar{x}) = g(\bar{x})$.

(b) Starting with the interval $[0, 1]$, use 4 iterations of the bisection method to calculate an approximation (an interval) for $\bar{x}$.

1) Pick midpoint $\frac{1}{2}$. $F(\frac{1}{2}) = -0.0487 < 0$ $\Rightarrow \bar{x} \in [0, \frac{1}{2}]$

2) Pick midpoint $\frac{1}{4}$. $F(\frac{1}{4}) = 0.598 > 0$ $\Rightarrow \bar{x} \in [\frac{1}{4}, \frac{1}{2}]$

3) Pick midpoint $\frac{3}{8}$. $F(\frac{3}{8}) = 0.298 > 0$ $\Rightarrow \bar{x} \in [\frac{3}{8}, \frac{1}{2}]$

4) Pick midpoint $\frac{7}{16}$ $F(\frac{7}{16}) = 0.1229 > 0$ $\Rightarrow \bar{x} \in [\frac{7}{16}, \frac{1}{2}]$

Answer:

First interval $\boxed{[0, 1/2]}$, Second interval $\boxed{[1/4, 1/2]}$

Third interval $\boxed{[\frac{3}{8}, \frac{1}{2}]}$, Fourth interval $\boxed{[\frac{7}{16}, \frac{1}{2}]}$

(c) Starting with $x_0 = 0$, use 4 iterations of Newton's method to calculate an approximation for $\bar{x}$.

Answer: The iteration for Newton's method is

$$x_{n+1} = \boxed{X_n - \frac{\frac{2}{1+x_n^2} - e^{x_n}}{-\frac{4x_n}{(1+x_n^2)^2} - e^{x_n}}}$$

$$F(x) = \frac{2}{1+x^2} - e^x, \qquad F'(x) = \frac{-4x}{(1+x^2)^2} - e^x$$

$$X_0 = 0$$

$$X_1 = 0 - \frac{1}{-1} = 1$$

$$X_2 = 1 - \frac{1-e}{-1-e} = 0.5379$$

$$X_3 = \; = 0.4843$$

$$X_4 = \; = 0.4833$$

$x_1 = \boxed{1}$

$x_2 = \boxed{0.5379}$

$x_3 = \boxed{0.4843}$

$x_4 = \boxed{0.4833}$