MAT 1330, Fall 2016 Assignment 5

Due Thursday November 10 by 8:00pm.

Late assignments will not be accepted; nor will unstapled assignments. Professors in the math department will not lend you a stapler; do not ask for one.

Please print this assignment double sided: to save paper and to ease weight.

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Student Name ______________________ Student Number ______________________

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Solutions

By signing below, we declare that this work is our own and that we have not copied from any other individual or other source.

Signatures ____

QUESTION 1. A company wants to build a cylindrical container with a semi-sphere lid. For a fixed volume V, the company wants to use a minimal amount of material for container and lid combined. Which radius r and height h of the container minimize the surface area for container and lid combined?

Answer: The optimal choice for r and h are $\boxed{r = \sqrt[3]{\frac{V}{3\pi}}, \quad h = \frac{V}{\pi\left(\frac{V}{3\pi}\right)^{2/3}}} = 3^{2/3} \cdot \left(\frac{V}{\pi}\right)^{1/3}$

- Volume $V = \pi r^2 h$ ①
- Surface Area $S = 2\pi r h + \pi r^2 + 2\pi r^2 = 2\pi r h + 3\pi r^2$ ②
- find h from ①: $h = \frac{V}{\pi r^2}$
- write ② in terms of r: $S(r) = \frac{2V}{r} + 3\pi r^2$.
- $S'(r) = -\frac{2V}{r^2} + 6\pi r$, $S'(r) = 0 \Rightarrow r = \sqrt[3]{\frac{V}{3\pi}}$.
- $S''(r) = \frac{4V}{r^3} + 6\pi > 0$ for all $r > 0$

$\Rightarrow$ S is concave up $\forall r > 0$

$\Rightarrow$ At $r = \sqrt[3]{\frac{V}{3\pi}}$ we have a local and global max.

QUESTION 2. Consider the function $f(x) = x^3 + 2x^2 + bx$.

1) The equation of the tangent line to the graph of this function at $x = 1$ is given by $y = \boxed{(7+b)x - 4}$

- $f'(x) = 3x^2 + 4x + b$
  $\Rightarrow f'(1) = 7+b$
- $f(1) = 3+b$ $\Rightarrow y = (7+b)x + c$
  $\Rightarrow 3+b = (7+b)(1) + c$
  $\Rightarrow c = -4$

2) The tangent line intersects the x-axis at $x = \boxed{\frac{4}{7+b}}$

$(7+b)x - 4 = 0 \Rightarrow x = \dfrac{4}{7+b}$

3) For what value(s) of b is there no intersection? Why not?

For $b = -7$ because the tangent line $y = -4$ is a horizontal line.

QUESTION 3. Find the distance of the line $y = 2x + 6$ from the circle $x^2 + y^2 = 0.5$ Find the point of the line that is closest to the circle.
[Hint: a point on the line is closest to the circle if it is closest to the centre of the circle.]

Answer: The distance is $\boxed{1.976 \text{ or } \frac{6}{5}\sqrt{5} - \sqrt{\frac{1}{2}}}$

- The distance between a point (x, y) on the line and the center of the circle $O(0,0)$ is: $d = \sqrt{x^2 + y^2} = \sqrt{x^2 + (2x+6)^2}$
  $= \sqrt{5x^2 + 24x + 36}$
- To find critical points, minimize $D(x) = 5x^2 + 24x + 36$.
  $\Rightarrow D'(x) = 10x + 24$, $D'(x) = 0 \Leftrightarrow x = \frac{-24}{10} = -2.4$
  $D''(x) = 10 > 0$ so $D(x)$ is concave up for all x
  $\Rightarrow$ At $x = -2.4$ we have a local and global min.
- For $x = -2.4$, $y = 2(-2.4) + 6 = 1.2$ $\Rightarrow$ Point $P(-2.4, 1.2)$ is the closest to the circle
- Distance from P to O $= d(-2.4) = \sqrt{7.2} = 2.683$
  $\Rightarrow$ Distance from P to the circle $= 2.683 - \sqrt{0.5} = 1.976$

Question 4. Consider the function $f(x) = e^{\sin(\cos(x))}$ on the interval $[0, 4\pi]$.

1) The critical points of f are at $\boxed{x = 0,\ x = \pi,\ x = 2\pi,\ x = 3\pi,\ x = 4\pi}$

[Hint: The range of the function $\cos(x)$ is $[-1, 1]$. What is the range of the function $\cos(\cos(x))$?]

$f'(x) = e^{\sin(\cos(x))} \cdot \cos(\cos(x)) \cdot (-\sin(x))$

$f'(x) = 0 \Rightarrow \sin x = 0 \Rightarrow x = 0, \pi, 2\pi, 3\pi, 4\pi$.

Note that $\cos(\cos(x)) > 0$ because its range is $[\underbrace{\cos(1)}_{>0}, 1]$.

2) Local maxima are at $\boxed{x = 2\pi}$

Local minima are at $\boxed{x = \pi,\ x = 3\pi}$

- The sign of $f'(x)$ is given by $-\sin(x)$:

	0		π		2π		3π		4π
f'		$-$		$+$		$-$		$+$	
f		↘		↗		↘		↗	

- Note that $x = 0$ and $x = 4\pi$ are not local max because they are endpoints of $[0, 4\pi]$.

3) Global maxima are at $\boxed{x = 0,\ 2\pi,\ 4\pi}$

Global minima are at $\boxed{x = \pi,\ 3\pi}$

4) Consider the same function, but now defined for all real numbers. Does it have a global maximum or minimum? Explain.

The function is periodic and has infinitely many global max and min (that are also local extrema).

$\Rightarrow$ Global max at points $x = 2\pi n$, $n \in \mathbb{Z}$

Global min at points $x = \pi + 2\pi n$, $n \in \mathbb{Z}$.

QUESTION 5. The DTDS $x_{t+1} = \frac{5x_t}{1+x_t} - hx_t$ with $h > 0$ gives the dynamics of an exploited fish stock. The first term on the right hand side is the net reproduction (according to a Beverton Holt function) and the second is the harvest. Calculate the steady state(s), x^*, of this DTDS. The yield at steady state is $Y^* = hx^*$, where x^* is the positive steady state, and it depends on h. Find the value h that maximizes the yield.

Answer: The optimal harvesting intensity is $h = \boxed{-1+\sqrt{5}}$.

- Steady States: $x^* = \frac{5x^*}{1+x^*} - hx^* \Rightarrow x^* = 0$ and $x^* = \frac{4-h}{1+h}$

In order for the second steady state to be positive, we need $0 < h < 4$.

- Yield: $Y^* = h.x^* = h\left(\frac{4-h}{1+h}\right)$

$$\Rightarrow Y^{*\prime} = \frac{-h^2-2h+4}{(1+h)^2}$$

$Y^{*\prime} = 0 \Rightarrow -h^2+2h+4 = 0 \Leftrightarrow h = \frac{-2 \pm \sqrt{4+16}}{2} = -1 \pm \sqrt{5}$

Since $0 < h < 4$, the only possible critical point is $h = -1+\sqrt{5} = 1.23$

	0		$-1+\sqrt{5}$	
$Y^{*\prime}$		$+$	0	$-$
Y^*		$\nearrow$		$\searrow$

$\Rightarrow$ At $h = 1.23$, $Y^{*\prime}$ goes from positive to negative, so Y^* has a local and global max there