Solution

MAT 1330, Fall 2016 Assignment 4

Due Thursday November 3 by 8:00pm.

Late assignments will not be accepted; nor will unstapled assignments. Professors in the math department will not lend you a stapler; do not ask for one.

Please print this assignment double sided: to save paper and to ease weight.

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QUESTION 1. For each of the following functions, find its derivative.

(a) $f(x) = \ln\left(\frac{(x-2)^7 e^{2x}}{(x+5)^8 e^{-9x^2}}\right)$

$$f(x) = \ln((x-2)^7 e^{2x}) - \ln((x+5)^8 e^{-9x^2})$$

$$= \ln(x-2)^7 + \ln e^{2x} - \ln(x+5)^8 - \ln e^{-9x^2} = 7\ln(x-2) + 2x - 8\ln(x+5) + 9x^2$$

$$f'(x) = \frac{7}{x-2} + 2 - \frac{8}{x+5} + 18x = \frac{7(x+5) - 8(x-2)}{(x-2)(x+5)} + \frac{2(x-2)(x+5) + 18x(x-2)(x+5)}{(x-2)(x+5)}$$

$$= \frac{7x+35-8x+16+2(x^2+3x-10)+18x(x^2+3x-10)}{(x-2)(x+5)} = \frac{18x^3+56x^2-175x+31}{(x-2)(x+5)}$$

Answer: $f'(x) = \boxed{\frac{18x^3+56x^2-175x+31}{(x-2)(x+5)}}$

(b) $g(x) = x^{\sin(\sqrt{x})}$

$g(x) = e^{\ln x^{\sin(\sqrt{x})}} = e^{\sin(\sqrt{x})\ln x}$, $g'(x) = e^{\sin(\sqrt{x})\ln x}$.

$[\sin(\sqrt{x})\ln x]' = e^{\sin\sqrt{x}\ln x} \cdot [(\sin(\sqrt{x}))' \ln x + \sin(\sqrt{x})(\ln x)']$

$= e^{\sin\sqrt{x}\ln x} \cdot [\cos(\sqrt{x}) \frac{1}{2\sqrt{x}} \cdot \ln x + \sin(\sqrt{x}) \cdot \frac{1}{x}] = x^{\sin(\sqrt{x})}[\frac{\cos(\sqrt{x})\ln x}{2\sqrt{x}} + \frac{\sin(\sqrt{x})}{x}]$

Answer: $g'(x) = \boxed{x^{\sin(\sqrt{x})}\left[\frac{\cos(\sqrt{x})\ln x}{2\sqrt{x}} + \frac{\sin(\sqrt{x})}{x}\right]}$

(c) $h(x) = \dfrac{\exp(x\ln(x))}{\ln(x\exp(x))}$

$$h'(x) = \frac{(e^{x\ln x})' \ln(xe^x) - e^{x\ln x}(\ln(xe^x))'}{(\ln(xe^x))^2}$$

$$= \frac{e^{x\ln x}(\ln x + 1)\ln(xe^x) - e^{x\ln x}\frac{1}{xe^x}(e^x + xe^x)}{(\ln(xe^x))^2}$$

$$= \frac{x^{x+1}(\ln x + 1)\ln(xe^x) - x^x(1+x)}{x(\ln(xe^x))^2}$$

Answer: $h'(x) = \boxed{\frac{x^{x+1}(\ln x + 1)\ln(xe^x) - x^x(1+x)}{x(\ln(xe^x))^2}}$

(d) $u(x) = \arctan(x)$

$\tan(\arctan x) = x$, Take derivative with respect to x at both sides:

$\sec^2(\arctan x)(\arctan x)' = 1$, $(\arctan x)' = \frac{1}{\sec^2(\arctan x)}$

$= \frac{1}{1 + \tan^2(\arctan x)} = \frac{1}{1 + (\tan(\arctan x))^2} = \frac{1}{1 + x^2}$

Answer: $u'(x) = \boxed{\frac{1}{1+x^2}}$

QUESTION 2. The equation $x^2y - xy^2 + e^{2x} + y^3 = 0$ implicitly defines a function $y(x)$.

(a) Find an expression for $y'(x)$ by implicit differentiation.

$$(2xy + x^2y') - (y^2 + x\cdot 2y\cdot y') + e^{2x}\cdot 2 + 3y^2y' = 0$$

$$2xy + x^2y' - y^2 - 2xyy' + 2e^{2x} + 3y^2y' = 0$$

$$y'(x^2 - 2xy + 3y^2) + 2xy - y^2 + 2e^{2x} = 0$$

$$y' = \frac{y^2 - 2xy - 2e^{2x}}{x^2 - 2xy + 3y^2}$$

Answer: $y'(x) =$ $\boxed{\dfrac{y^2 - 2xy - 2e^{2x}}{x^2 - 2xy + 3y^2}}$

(b) Calculate the value $y(0)$ and find the equation of the tangent line to the function at the point $(0, y(0))$.

$x = 0$, then $x^2y - xy^2 + e^{2x} + y^3 = 0 \Rightarrow 1 + y^3 = 0 \Rightarrow y = -1$

Therefore $y(0) = -1$

$$y'(0) = \frac{(-1)^2 - 2}{3(-1)^2} = \frac{-1}{3} = -\frac{1}{3}$$

Tangent line: $y - (-1) = -\frac{1}{3}(x - 0)$

$$y + 1 = -\frac{1}{3}x$$

$$y = -\frac{1}{3}x - 1$$

Answer: The equation of the line is $y =$ $\boxed{-\frac{1}{3}x - 1}$

QUESTION 3. Consider the function

$$f(x) = \frac{x+1}{x} e^x.$$

(a) The domain of f is $\boxed{x \neq 0}$

(b) $\lim_{x\to\infty} f(x) = \boxed{\infty}$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{(x+1)e^x}{x} = \lim_{x\to\infty} \left(1+\frac{1}{x}\right)e^x = \infty$$

(c) $\lim_{x\to-\infty} f(x) = \boxed{0}$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{(x+1)e^x}{x} = \lim_{x\to-\infty} \left(1+\frac{1}{x}\right)e^x$$

$$= \lim_{x\to-\infty} \left(1+\frac{1}{x}\right) \cdot \lim_{x\to-\infty} e^x = 1 \cdot 0 = 0$$

(d) Find the (one-sided) limit(s) of f as x approaches any number not in the domain of f.

$$\lim_{x\to0^+} f(x) = \lim_{x\to0^+} \frac{(x+1)e^x}{x} = \lim_{x\to0^+} \left(1+\frac{1}{x}\right)e^x = \infty$$

$$\lim_{x\to0^-} f(x) = \lim_{x\to0^-} \left(1+\frac{1}{x}\right)e^x = -\infty$$

(e) The zeros of f are $\boxed{-1}$

$(x+1)e^x = 0 \Rightarrow x = -1$

(f) The derivative of f is $f'(x) = \boxed{\frac{e^x}{x^2}(x^2+x-1)}$

$f'(x) = \left(\frac{x+1}{x}\right)' e^x + \left(\frac{x+1}{x}\right) e^x$

$= \frac{x-(x+1)}{x^2} e^x + \left(\frac{x+1}{x}\right) e^x = -\frac{1}{x^2} e^x + \frac{x+1}{x} e^x$

$= \frac{e^x}{x^2}(x^2+x-1)$

(g) The critical points of f are $\boxed{\frac{-1 \pm \sqrt{5}}{2}}$

$f'(x) = 0 \Rightarrow x^2+x-1 = 0 \Rightarrow x = \frac{-1 \pm \sqrt{5}}{2}$

(h) f is increasing in the interval(s) $\boxed{\left(-\infty, \frac{-1-\sqrt{5}}{2}\right) \cup \left(\frac{-1+\sqrt{5}}{2}, \infty\right)}$

$f'(x) > 0 \Rightarrow x^2+x-1 > 0 \Rightarrow x > \frac{-1+\sqrt{5}}{2}$

or $x < \frac{-1-\sqrt{5}}{2}$

(i) f is decreasing in the interval(s) $\boxed{\left(\frac{-1-\sqrt{5}}{2}, 0\right) \cup \left(0, \frac{-1+\sqrt{5}}{2}\right)}$

$f'(x) < 0 \Rightarrow x^2+x-1 < 0 \Rightarrow \frac{-1-\sqrt{5}}{2} < x < \frac{-1+\sqrt{5}}{2}$

But 0 is not in the domain.

(j) The second derivative of f is $f''(x) = \boxed{\dfrac{e^x(x^3+x^2-2x+2)}{x^3}}$

$$f''(x) = \left(\frac{e^x}{x^2}\right)'(x^2+x-1) + \frac{e^x}{x^2}(2x+1)$$

$$= \frac{e^x x^2 - e^x 2x}{x^4}(x^2+x-1) + \frac{e^x}{x^2}(2x+1)$$

$$= \frac{e^x(x-2)(x^2+x-1)}{x^3} + \frac{e^x(2x^2+x)}{x^3}$$

$$= \frac{e^x[x^3+x^2-2x+2]}{x^3}$$

(k) It is not easy to calculate the zeros of f''. Use the sign (change) of the first derivative to determine which critical points are minima or maxima.

f has a maximum at $\boxed{\frac{-1-\sqrt{5}}{2}}$

f has a minimum at $\boxed{\frac{-1+\sqrt{5}}{2}}$

You may use the fact that the only zero of the expression $x^3 + x^2 - 2x + 2$ is located at $x \approx -2.268$. You do not have to show that this is the only zero.

Critical points: $x_1 = \frac{-1-\sqrt{5}}{2}$, $x_2 = \frac{-1+\sqrt{5}}{2}$

$x_1 \approx -1.6$ $\quad x_2 \approx 0.62$

Because $x \to \infty$, $x^3+x^2-2x+2 \to \infty$, $x \approx -2.268$ is the only zero, $x_1^3+x_1^2-2x_1+2 > 0$, $x_2^3+x_2^2-2x_2+2>0$.

Therefore $f''(x_1) < 0$, $f''(x_2) > 0$

(1) Sketch the graph of the function on the interval $[-5, 5]$

$f''(x) \leq 0$ if $-2.268 \leq x < 0$

$f''(x) \geq 0$ if $-5 \leq x \leq -2.268$ or $0 < x \leq 5$.