MAT 1330, Fall 2016 Assignment 2

Due Thursday September 29 by 8:00pm.

Late assignments will not be accepted; nor will unstapled assignments. Professors in the math department will not lend you a stapler; do not ask for one.

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By signing below, we declare that this work is our own and that we have not copied from any other individual or other source.

Signatures ______________________________

QUESTION 1. (a) Suppose that a cubic polynomial $x^3 + Ax^2 + Bx + C$ factors into $(x-a)(x-b)(x-c)$ where x is a variable and a, b, c are real numbers. Calculate the expressions of A, B, C in terms of a, b, c.

$A = \boxed{-(a+b+c)}$

$B = \boxed{a(b+c)+bc}$

$C = \boxed{-abc}$

$$(x-a)(x-b)(x-c) = (x-a)\left(x^2 - x(b+c) + bc\right)$$
$$= x^3 - x^2(b+c) + bcx - ax^2 + ax(b+c) - abc$$
$$= x^3 - x^2(b+c+a) + x\left(bc + a(b+c)\right) - abc$$

(b) Find all the roots of $f(x) = x^3 - 4x^2 - 11x + 30$.

Answer: $\boxed{(x-2)(x-5)(x+3)}$

One factor of $f(x)$ is 2 since $f(2) = 2^3 - 4(2)^2 - 11(2) + 30 = 0$. Therefore

$f(x) = (x-2)(x-b)(x-c)$

using part (a) we have

$-4 = -2 - (b+c) \Rightarrow b = 2 - c$ ①

$-11 = 2(b+c) + bc$ ②

$30 = -2bc$ ③

$\hookrightarrow b = \frac{-15}{c}$

plug ① into ②: $-11 = 2(2 - c + c) + (2-c)c$

$-11 = 4 + 2c - c^2$

P: -15 -5×3

S: -2

$c^2 - 2c - 15 = 0$

$(c-5)(c+3) = 0$

$c = 5$ or $c = -3$

But we need this to satisfy ③ so

for $c = 5$: $b = \frac{-15}{5} = -3$

and

$c = -3$: $b = \frac{-15}{-3} = 5$

QUESTION 2. In each of the following expressions, find all possible solutions for x.

(a) $\log_{10}(100x^3) = 4$.

Answer: $x = \boxed{\sqrt[3]{100} = 10^{2/3}}$

$\log_{10} 100 + \log_{10} x^3 = 4$

$\log_{10} 10^2 + \log_{10} x^3 = 4$

$2\log_{10} 10 + \log_{10} x^3 = 4$

$2 + \log_{10} x^3 = 4$

$\log_{10} x^3 = 2$

$x^3 = 10^2$

$x^3 = 100$

$x = \sqrt[3]{100} = 10^{2/3}$

(b) $\frac{ax}{b+x} + cx = dx$.

Answer: $x = \boxed{\emptyset \text{ if } c=d \;\&\; a\neq 0;\; x=\mathbb{R} \text{ if } c=d \text{ and } a=0;\; x=0 \;\&\; x=\frac{db-bc-a}{c-d} \text{ if } c\neq d}$

$\frac{ax}{b+x} + (c-d)x = 0$

$x \neq -b$

$\frac{ax + (c-d)x(b+x)}{b+x} = 0$

$ax + x(c-d)(b+x) = 0$

$x(a + (c-d)(b+x)) = 0$

$\underline{\underline{x=0}}$ or

$a + cb - xd + xc - db = 0$

$x(c-d) = db - cb - a$

Case ①: if $c=d$ and $a=0$ then all $\mathbb{R}$ are solutions

case ②: if $c=d$ and $a\neq 0$, then $x=0$ is the only solution

case ③. if $c\neq d$, then

$x = \frac{db-bc-a}{c-d}$

(c) $\log_{10}(x) + \log_{10}(x-3) = 1$.

Answer: $x = \boxed{5}$

$\log_{10}(x(x-3)) = 1$

$x^2 - 3x = 10^1$

$x^2 - 3x - 10 = 0$

$(x-5)(x+2) = 0$

$x = 5$ or $x = -2$

impossible since $\log(-2)$ does not exist

Question 3.

The number of fish in a lake decreases by 25% each year since anglers take out more than fish reproduce. To prevent a collapse of the population, the municipality decides to restock 500 fish at the end of each year. The DTDS for the number of fish in the lake in year t, denoted by x_t, is given by $x_{t+1} = 0.75x_t + 500$.

(a) The updating function of the DTDS is $f(x) = \boxed{0.75x+500}$

(b) The general solution of the DTDS is $x_t = \boxed{0.75^t(X_0-2000)+2000}$

(c) The steady state of the DTDS is $x^* = \boxed{2000}$

$x^* = \frac{c}{1-r} = \frac{500}{1-0.75} = 2000$ || or another way to find x^*:

$x^* = 0.75x^*+500$

$x^* - 0.75x^* = 500$

$0.25x^* = 500 \Rightarrow x^* = 2000$

(d) Early one year, after a particularly cold and long winter, the municipality counts only $x_0 = 800$ fish in the lake. Calculate x_1, x_2, x_3.

$X_0 = 800$

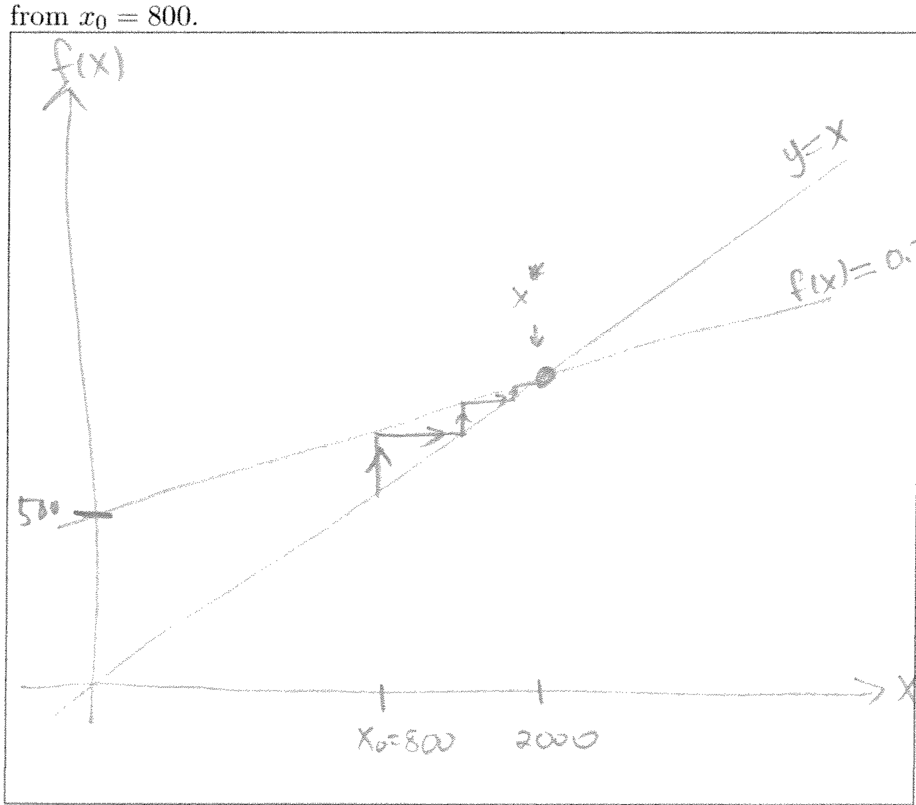
$X_1 = 0.75(800)+500 = 1100$

$X_2 = 0.75(1100)+500 = 1325$

$X_3 = 0.75(1325)+500 = 1493.75$

Answer $\boxed{X_1 = 1100,\ X_2 = 1325,\ X_3 = 1493.75}$ or $X_3 = 1493$ fish

(e) Draw the graph of the updating function and cobweb the solution of the DTDS starting from $x_0 = 800$.



(f) Is the steady state stable? Justify your answer through **two** different ways of reasoning.

The steady state is stable because;
① The solution in the cobweb shows that it is going towards the steadystate
② The value of r in $X_{t+1} = rX_t + C$ is $|r| = |0.75| = 0.75 < 1$, meaning the steady state is stable

(g) How long will it take for a population of $x_0 = 800$ fish to grow up to 80% of the steady state value?

Answer: 4 years

We want our population X_t to reach 80% of 2000 ($= x^*$), so 1600. We need to solve for t when $X_t = 1600$.

$1600 = 0.75^t(800 - 2000) + 2000$

$-400 = 0.75^t(-1200)$

$0.333 = 0.75^t$

$\ln 0.333 = \ln(0.75^t)$

$\ln(0.333) = t \ln(0.75)$

$t = \frac{\ln 0.333}{\ln(0.75)} = 3.82$

∴ It will take 4 years.

(h) The municipality wants to increase the steady state level of fish in the lake to 3000, but they cannot afford a larger restocking program. Therefore, they decide to limit fishing. What is the percentage of annual decrease (in percent) that they can allow to get the steady state to be 3000?

16.67%

We want $X^* = 3000$. → but we can't increase the "restock" $X_{t+1} = 0.75X_t + 500$

So we want to find r in $X_{t+1} = rX_t + 500$ in order to have $X^* = 3000$

$X^* = rX^* + 500$

$rX^* = X^* - 500$

$r = \frac{X^* - 500}{X^*}$

$r = \frac{3000 - 500}{3000} = 0.833$

Thus, instead of having an annual decrease of 25% each year (this gives $X^* = 2000$), we need to limit this to an annual decrease of 16.67%

QUESTION 4. The plot below shows you the graph of an updating function $y = f(x)$ and the diagonal $y = x$.

(a) How many steady states does the DTDS $x_{t+1} = f(x_t)$ have and what are their approximate values? Mark them in the plot.

Answer: 3 steady states; $x_1^* = 0.1$, $x_2^* = 0.35$, $x_3^* = 0.7$

(b) Start the cobwebbing process at the value $x_0 = 0.3$. What is the long-term behavior of the corresponding solution?

Answer: population equilibrates at $x_1^* = 0.1$

(c) Start the cobwebbing process at the value $x_0 = 0.5$. What is the long-term behavior of the corresponding solution?

Answer: population equilibrates at $x_1^* = 0.1$

(d) Based on your cobwebbing, indicate which of the steady states seem to be stable and which seem unstable.

Answer: $x_1^* = 0.1$, stable; $x_2^* = 0.35$, unstable; $x_3^* = 0.7$, unstable