MATH 209/4 all sections except EC: - Fundamental Mathematics II

Midterm - March 5, 2017 (1h30min)
Only approved calculators are permitted.

MARKS

[7] **1.** (a) Find $\lim_{x\to 0} \frac{x}{\sqrt{x+49}-7}$.

[7] (b) Give functions $f(x)$ and $g(x)$ with the following properties:

(i) $\lim_{x\to 5} f(x) = 0$ (ii) $\lim_{x\to 5} g(x) = 0$ (iii) $\lim_{x\to 5} \frac{f(x)}{g(x)} = \frac{1}{5}$

[7] **2.** Let $h(x) = 6 - x^3$. Work out the following in detail:

$$\lim_{s\to 0} \frac{[h(x+s) - h(x)]}{s}$$

[12] **3.** (a) If $f(x) = -3x^{27} - 25$, find $f'(x)$. Do not simplify.

(b) If $g(x) = (5x^3 - 4)[ln(x^2) + 2]$, find $g'(x)$. Do not simplify.

(c) Find $h'(x)$ if $h(x) = \frac{x^3 - e^x}{[e^{3x} + ln(x)]}$. Do not simplify.

(d) Find the value of dy if $y = x^4 + 2, x = 3$ and the change in x is 0.2.

[7] **4.** A stock grows from ten dollars to twenty dollars in nine years. Find the associated annual rate of growth assuming that it is compounded continuously.

[7] **5.** Suppose that a cost function is given by $C(x) = 8,000 + 7x$.

A student produced the following argument for finding the marginal average cost function: $C'(x) = 7$, $\frac{C'(x)}{x} = \frac{7}{x}$. Why is this answer incorrect?

[10] **6.** A point is moving on the graph of $y^3 = x^2$. When the point is at $(-8, 4)$, its y-coordinate is increasing by 3 units per second. How fast is the x-coordinate changing at that moment?

[13] **7.** Find the equation(s) of the tangent line(s) to the graph of $y - xy^2 + x^2 + 1 = 0$ at the point(s) where $x = 1$.

1. a) Find $\lim_{x \to 0} \frac{x}{\sqrt{x+49} - 7}$

A) $\lim_{x \to 0} \frac{x}{\sqrt{x+49} - 7} = \frac{0}{\sqrt{49} - 7} = \frac{0}{0}$ Indeterminate form

We have to simplify algebraically.

$$\lim_{x \to 0} \frac{x}{\sqrt{x+49} - 7} = \lim_{x \to 0} \frac{x}{\sqrt{x+49} - 7} \times \frac{\sqrt{x+49} + 7}{\sqrt{x+49} + 7}$$

$$= \lim_{x \to 0} \frac{x\left(\sqrt{x+49} + 7\right)}{(x+\cancel{49}) - \cancel{49}} = \lim_{x \to 0} \frac{\cancel{x}\left(\sqrt{x+49} + 7\right)}{\cancel{x}}$$

$$= \lim_{x \to 0} \sqrt{x+49} + 7 = \sqrt{49} + 7 = 14.$$

b) Give functions $f(x)$, $g(x)$ with properties

i) $\lim_{x \to 5} f(x) = 0$ ii) $\lim_{x \to 5} g(x) = 0$ iii) $\lim_{x \to 5} \frac{f(x)}{g(x)} = \frac{1}{5}$

One example : $f(x) = x - 5 \Longrightarrow \lim_{x \to 5} f(x) = (5 - 5) = 0$

$g(x) = 5(x-5) \Rightarrow \lim_{x \to 5} g(x) = 5(5-5) = 0$

$$\lim_{x \to 5} \frac{f(x)}{g(x)} = \lim_{x \to 5} \frac{\cancel{(x-5)}}{5\cancel{(x-5)}} = \frac{1}{5}$$

2. Let $h(x) = 6 - x^3$. Work out

$$\lim_{s \to 0} \frac{[h(x+s) - h(x)]}{s}$$

$$h(x+s) = 6 - (x+s)^3 = 6 - x^3 - s^3 - 3x^2s - 3xs^2$$

$$\lim_{s \to 0} \frac{[h(x+s) - h(x)]}{s} = \lim_{s \to 0} \frac{[\cancel{6} - \cancel{x^3} - s^3 - 3x^2s - 3xs^2 - \cancel{6} + \cancel{x^3}]}{s}$$

$$\lim_{s \to 0} \frac{-\cancel{s}(s^2 + 3x^2 + 3xs)}{\cancel{s}}$$

$$\lim_{s \to 0} -(s^2 + 3x^2 + 3xs) = -3x^2$$

3. a) $f(x) = -3x^{27} - 25$

$$f'(x) = -3.27\,x^{26} = -81x^{26}$$

b) $g(x) = (5x^3 - 4)[\ln(x^2) + 2]$

$$g'(x) = 15x^2[\ln(x^2) + 2] + (5x^3 - 4)\frac{2}{x}.$$

c) $h(x) = \dfrac{x^3 - e^x}{e^{3x} + \ln x}$

$$h'(x) = \frac{(3x^2 - e^x)(e^{3x} + \ln x) - (x^3 - e^x)\left(3e^{3x} + \frac{1}{x}\right)}{(e^{3x} + \ln x)^2}.$$

d) $y = x^4 + 2$, $x = 3$, $dx = 0.2$, $dy = ?$

$dy = 4x^3\,dx$

$dy = 4(3)^3 \cdot 0.2 = (4)\cdot(27)\cdot(0.2) = 21.6$

4. A stock grows from \$10 to \$20 in 9 years. Find the associated annual rate of growth (compounded continuously).

4) $A = Pe^{rt}$

A = Final (total) amount
P = Principal
r = annual rate (compounded continuously)
t = time

$$20 = 10e^{r \cdot 9}$$

$$\frac{20}{10} = e^{r \cdot 9} \Rightarrow 2 = e^{9 \cdot r}$$

Taking ln (natural logarithm) on both sides

$$\ln 2 = \ln(e^{9 \cdot r}) = 9 \cdot r$$

$$\Rightarrow r = \frac{\ln 2}{9} = 0.077 = 7.7\% .$$

5) Cost function : $C(x) = 8{,}000 + 7x$.

Average cost function : $\overline{C}(x) = \frac{C(x)}{x} = \frac{8{,}000 + 7x}{x}$

$$= \frac{8000}{x} + 7$$

Marginal average cost : $\overline{C}'(x) = \left(\frac{C(x)}{x}\right)' = \left(\frac{8000}{x} + 7\right)'$

$$= \frac{-8000}{x^2}$$

The student is wrong because the order of the operations of taking average and differentiating matter.

Taking average and differentiating is not equal to differentiating first and then taking the average (as the student incorrectly did).

6 A point is moving on the graph of $y^3 = x^2$. When it is at $(-8, 4)$ the y-coordinate is increasing by 3 units/sec. How fast is the x-coordinate changing?

4) Given : $\frac{dy}{dt} = 3$ (3 units/sec $\Rightarrow$ rate of change of y-coordinate with respect to TIME).

Since the point is on the graph of $y^3 = x^2$, the ~~dex~~ rate of change of x and y-coordinates are related as.

$$\frac{d}{dt}(y^3) = \frac{d}{dt}(x^2)$$

$$\Rightarrow \quad 3y^2 \cdot \frac{dy}{dt} = 2x\frac{dx}{dt}$$

$$\text{or} \quad \frac{dx}{dt} = \frac{3y^2\frac{dy}{dt}}{2x} = \frac{3\overset{y^2}{(4)^2}\overset{dy/dt}{(3)}}{2\underset{x}{(-8)}}$$

$$= -9 \text{ units/sec.}$$

7. Find the equation(s) of tangents to the grap of $y - xy^2 + x^2 + 1 = 0$. at the point where $x = 1$.

1) Find the points P on the graph corresponding to $x = 1$.

2) Find the slope of the ~~[illegible]~~ $\left(m = \frac{dy}{dx}\right)$ tangent line.

3) Evaluate slope at the point(s) P found in step 1.

4) Find equations of the tangent lines passing through P.

1.) When $x = 1$ on the graph of $y - xy^2 + x^2 + 1$, we have

$$y - \underset{\substack{\uparrow \\ x}}{(1)}y^2 + \underset{\substack{\uparrow \\ x^2}}{(1)^2} + 1 = 0$$

$\therefore$ The corresponding values of the y-coordinate is (are) found by solving for y in the equation

$$y - y^2 + 2 = 0$$

$$y^2 - y - 2 = 0 \Rightarrow (y-2)(y+1) = 0$$

$$\Rightarrow \quad y = 2 \text{ or } y = -1.$$

The points corresponding to $x = 1$ are

$P_1 \equiv (1, 2)$, $P_2 \equiv (1, -1)$.

2.) Find $m = dy/dx$.

Differentiating both sides of $y - xy^2 + x^2 + 1 = 0$ w.r.t x, we get

$$\frac{d}{dx}(y) - \frac{d}{dx}(xy^2) + \frac{d}{dx}(x^2) + \frac{d}{dx}(1) = \frac{d}{dx}(0)$$

$$\Rightarrow \frac{dy}{dx} - \left\{ y^2 + x \cdot 2y \cdot \frac{dy}{dx} \right\} + 2x = 0 .$$

$$\Rightarrow \frac{dy}{dx} - y^2 - 2xy\frac{dy}{dx} + 2x = 0$$

or $$\frac{dy}{dx}(1 - 2xy) = y^2 - 2x .$$

or $$\frac{dy}{dx} = \frac{y^2 - 2x}{1 - 2xy} .$$

3.) Find $m = \frac{dy}{dx}$ at P_1 and P_2

m at P_1 : $$\left.\frac{dy}{dx}\right|_{(x,y)\equiv(1,2)} = \left.\frac{y^2 - 2x}{1 - 2xy}\right|_{(x,y)\equiv(1,2)}$$

$$= \frac{(2)^2 - 2(1)}{1 - 2(1)(2)} = -\frac{2}{3} .$$

m at P_2 : $\left.\dfrac{dy}{dx}\right|_{(x,y)\equiv(1,-1)} = \left.\dfrac{y^2-2x}{1-2xy}\right|_{(x,y)\equiv(1,-1)}$

$$= \frac{(-1)^2-2(1)}{1-2(1)(-1)} = \frac{-1}{3}.$$

4.) Find equations of tangent lines at P_1, P_2.

Eq of tangent at $P_1 \equiv (1,2)$ with slope $m = \frac{-2}{3}$.

$(y-y_1) = m(x-x_1)$ $\boxed{\text{OR}}$ $y = mx + b$

$= (y-2) = \frac{-2}{3}(x-1)$

equivalently written as

$$y = -\frac{2}{3}x + \frac{2}{3} + 2$$

$$\boxed{y = \frac{-2}{3}x + \frac{8}{3}}$$

$y = -\frac{2}{3}x + b$

To find b, we substitute $(x,y) \equiv (1,2)$

$2 = -\frac{2}{3}(1) + b$

or $b = 2 + \frac{2}{3} = \frac{8}{3}$

$\Rightarrow$ $\boxed{y = \frac{-2}{3}x + \frac{8}{3}}$

USE method OR method.

Eqn of ~~of~~ tangent at $P_2 \equiv (1, -1)$ with $m = -\frac{1}{3}$

$$(y - y_1) = m(x - x_1)$$

$$(y - (-1)) = -\frac{1}{3}(x - 1)$$

or $(y+1) = -\frac{1}{3}(x-1)$

Equivalently written as

$$y = -\frac{1}{3}x + \frac{1}{3} - 1$$

$$\boxed{y = -\frac{1}{3}x - \frac{2}{3}}$$