December 2015

Answer 1:

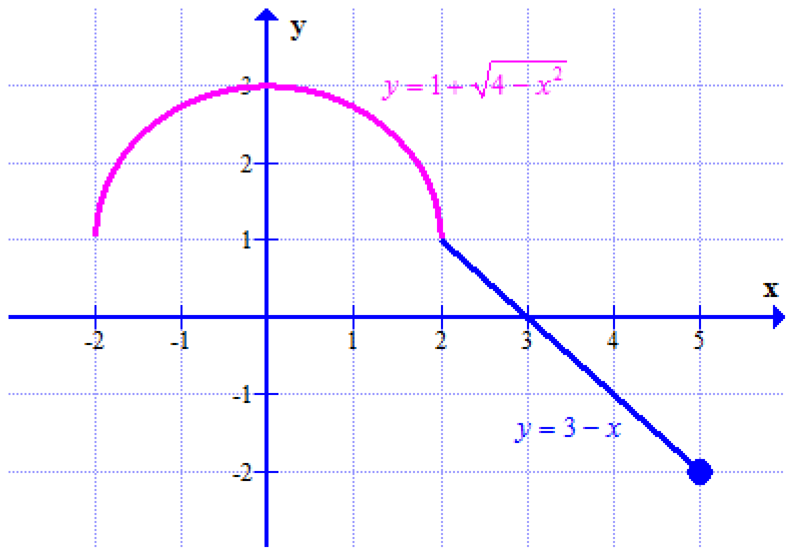
a)
Consider the given function

$$f(x)=\begin{cases}1+\sqrt{4-x^2}, & -2\le x\le 2\\ 3-x, & x>2\end{cases}$$

To sketch the graph $y=1+\sqrt{4-x^2}$ in between $x=-2$ and $x=2$, first sketch the upper half of semicircle having center at $(0,0)$ and radius 2, then shift it 1 unit up.

Now the graph of $y=3-x$ is a straight line passing though $(2,1)$ and $(3,0)$. The graph should be sketched only for x which is larger than 2.

Sketch the graph of $f(x)$ in interval $[-2,5]$:



Find the definite integral as

$$\int_{-2}^{5} f(x)\,dx=\int_{-2}^{2} f(x)\,dx+\int_{2}^{3} f(x)\,dx+\int_{3}^{5} f(x)\,dx$$

$$=\left[A_{semicircle}(r=2)+A_{\text{rectangle}}(4\times1)\right]+A_{triangle}(b=1,h=1)-A_{triangle}(b=2,h=2)$$

$$=\left[\frac{1}{2}\pi\cdot2^2+4\cdot1\right]+\frac{1}{2}\cdot1\cdot1-\frac{1}{2}\cdot2\cdot2$$

$$=2\pi+4+\frac{1}{2}-2$$

$$=\boxed{2\pi+\frac{5}{2}}$$

b)
Using Fundamental Thorem of Calculus

$$F'(x)=\frac{d}{dx}\left[\int_{-x^2}^{x} e^{1-t^2}\,dt\right]$$

$$=e^{1-x^2}\cdot\frac{d}{dx}(x)-e^{1-(-x^2)^2}\cdot\frac{d}{dx}(-x^2)$$

$$=e^{1-x^2}\cdot1-e^{1-x^4}\cdot(-2x)$$

$$=\boxed{e^{1-x^2}+2xe^{1-x^4}}$$

Now find $F'(1)$, replacing x by 1 to obtain

$$F'(1)=e^{1-1^2}+2\cdot1\cdot e^{1-1^4}$$

$$=1+2$$

$$=3$$

Since $F'(1)>0$, the $F(x)$ is $\boxed{\text{increasing}}$

Answer 2:

a)
Consider the given integral

$$\int\frac{\cos^3 x}{\sin^2 x}dx=\int\frac{\cos^2 x}{\sin^2 x}\cdot\cos x\,dx$$

$$=\int\frac{(1-\sin^2 x)}{\sin^2 x}\cdot\cos x\,dx$$

Let $\sin x=u$ then $\cos x\,dx=du$, hence integral becomes

$$\int\frac{\cos^3 x}{\sin^2 x}dx=\int\frac{(1-u^2)}{u^2}du$$

$$=\int\left(\frac{1}{u^2}-1\right)du$$

$$=-\frac{1}{u}-u+C$$

$$=\boxed{-\frac{1}{\sin x}-\sin x+C}\qquad[u=\sin x]$$

b)
Consider the given integral

$$\int\left(e^x+\ln x\right)dx=\int e^x dx+\int \ln x dx$$
$$=e^x+\int \ln x dx$$

Let

$$u=\ln x \qquad dv=dx$$
$$du=\frac{1}{x}dx \qquad v=\int dx=x$$

Hence using integral by parts $\int u dv=uv-\int v du$ formula

$$\int\left(e^x+\ln x\right)dx=e^x+\int \ln x dx$$
$$=e^x+\left[\ln x\cdot x-\int x\cdot\frac{1}{x}dx\right]$$
$$=e^x+x\ln x-\int dx$$
$$=\boxed{e^x+x\ln x-x+C}$$

Answer 3:

Find $F(x)$ as

$$F(x)=\int F'(x)dx$$
$$=\int\frac{x^2+4}{x^2-4}dx$$
$$=\int\left(1+\frac{8}{x^2-4}\right)dx$$
$$=\int dx+8\int\frac{1}{x^2-4}dx$$

Using the formula $\int\frac{1}{x^2-a^2}dx=\frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|$, it becomes

$$F(x)=x+8\cdot\frac{1}{2\cdot 2}\ln\left|\frac{x-2}{x+2}\right|+C$$
$$=x+2\ln\left|\frac{x-2}{x+2}\right|+C$$

Using the given condition $F(-1)=0$, find constant C as

$$F(-1)=-1+2\ln\left|\frac{-1-2}{-1+2}\right|+C$$
$$0=-1+2\ln 3+C$$
$$C=1-2\ln 3$$

Using $C=1-2\ln 3$, the equation $F(x)=x+2\ln\left|\frac{x-2}{x+2}\right|+C$ becomes

$$\boxed{F(x)=x+2\ln\left|\frac{x-2}{x+2}\right|+1-2\ln 3}$$

Answer 4:

a)
Consider the given integral

$$\int_0^{\frac{\pi}{2}}\frac{\cos x}{4+\sin^2 x}dx$$

Let $\sin x=u$ then $\cos x dx=du$,

When $x=0$ then $u=0$

When $x=\frac{\pi}{2}$ then $u=1$

$$\int_0^{\frac{\pi}{2}}\frac{\cos x}{4+\sin^2 x}dx=\int_0^1\frac{du}{4+u^2}$$
$$=\left[\frac{1}{2}\tan^{-1}\left(\frac{u}{2}\right)\right]_0^1$$
$$=\frac{1}{2}\tan^{-1}\left(\frac{1}{2}\right)-\frac{1}{2}\tan^{-1}(0)$$
$$=\frac{1}{2}\tan^{-1}\left(\frac{1}{2}\right)-0$$
$$=\boxed{\frac{1}{2}\tan^{-1}\left(\frac{1}{2}\right)}$$

b)
Consider the given integral

$$\int_0^{\frac{\pi}{4}} \sec^4 x dx = \int_0^{\frac{\pi}{4}} \sec^2 x \cdot \sec^2 x dx$$

$$= \int_0^{\frac{\pi}{4}} \left(1+\tan^2 x\right)\cdot \sec^2 x dx \qquad \left[\sec^2 x - \tan^2 x = 1\right]$$

Let $\tan x = u$ then $\sec^2 x dx = du$,

When $x = 0$ then $u = 0$

When $x = \frac{\pi}{4}$ then $u = 1$

$$\int_0^{\frac{\pi}{4}} \sec^4 x dx = \int_0^1 \left(1+u^2\right) du$$

$$= \left[u + \frac{u^3}{3}\right]_0^1$$

$$= 1 + \frac{1}{3} - 0$$

$$= \boxed{\frac{4}{3}}$$

c)

Consider the given integral

$$\int_0^3 x^2 \sqrt{1+x} dx$$

Let $1 + x = u \Rightarrow x = u - 1$ then $dx = du$,

When $x = 0$ then $u = 1$

When $x = 3$ then $u = 4$

$$\int_0^3 x^2 \sqrt{1+x} dx = \int_1^4 (u-1)^2 \sqrt{u} du$$

$$= \int_1^4 \left(u^2 - 2u + 1\right) u^{\frac{1}{2}} du$$

$$= \int_1^4 \left(u^{\frac{5}{2}} - 2u^{\frac{3}{2}} + u^{\frac{1}{2}}\right) du$$

$$= \left[\frac{2}{7}u^{\frac{7}{2}} - 2\cdot\frac{2}{5}u^{\frac{5}{2}} + \frac{2}{3}u^{\frac{3}{2}}\right]_1^4$$

$$= \left(\frac{2}{7}\cdot 128 - \frac{4}{5}\cdot 32 + \frac{2}{3}\cdot 8\right) - \left(\frac{2}{7} - \frac{4}{5} + \frac{2}{3}\right)$$

$$= \boxed{\frac{1696}{105}}$$

Answer 5:

a)

Consider the given integral

$$\int_e^\infty \frac{1}{x \ln^3 x} dx = \int_e^\infty \frac{1}{\ln^3 x} \cdot \frac{1}{x} dx$$

Let $\ln x = u$ then $\frac{1}{x} dx = du$,

When $x = e$ then $u = 1$

When $x \to \infty$ then $u \to \infty$

$$\int_e^\infty \frac{1}{x \ln^3 x} dx = \int_1^\infty \frac{du}{u^3}$$

$$= \lim_{t\to\infty} \int_1^t \frac{du}{u^3}$$

$$= \lim_{t\to\infty} \left[-\frac{1}{2u^2}\right]_1^t$$

$$= \lim_{t\to\infty} \left(\frac{1}{2} - \frac{1}{2t^2}\right)$$

$$= \frac{1}{2} - 0$$

$$= \boxed{\frac{1}{2}}$$

b)

Consider the given integral

$$\int_{-1}^1 \frac{xdx}{x^2 - 1}$$

This integral is discontinuous at both limits $x = \pm 1$ and hence we will split the integral up at any point of our choice which is convenient to evaluate

$$\int_{-1}^1 \frac{xdx}{x^2-1} = \int_{-1}^0 \frac{xdx}{x^2-1} + \int_0^1 \frac{xdx}{x^2-1}$$

Now we will look at each of these integrals and see if they are convergent,

$$\int_0^1 \frac{xdx}{x^2-1} = \lim_{t\to 1^-}\int_0^t \frac{xdx}{x^2-1}$$

$$= \lim_{t\to 1^-}\left[\frac{1}{2}\ln\left|x^2-1\right|\right]_0^t \qquad \left[x^2-1=u \Rightarrow xdx=\frac{1}{2}du\right]$$

$$= \frac{1}{2}\lim_{t\to 1^-}\left[\ln\left|t^2-1\right|-\ln 1\right]$$

$$= \frac{1}{2}(-\infty-0)$$

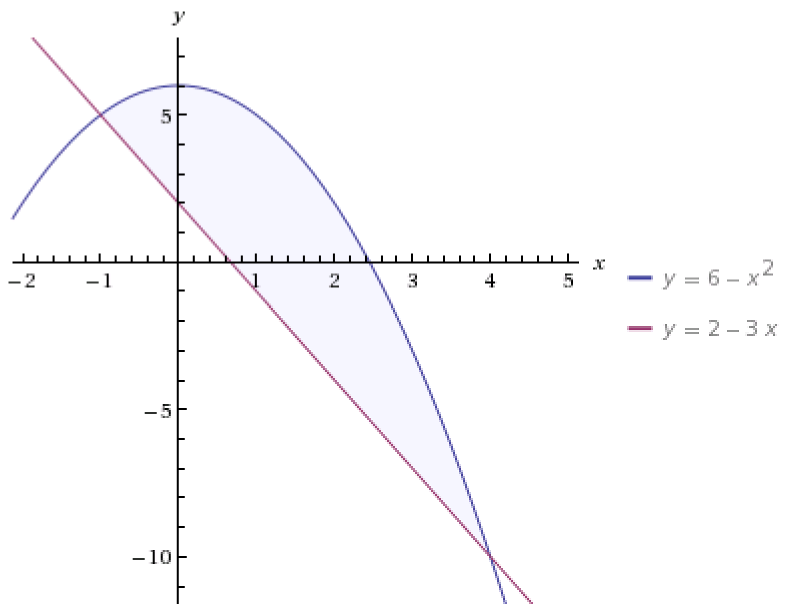
$$= -\infty$$

Since one of the integral is divergent, we don't need to check other. Hence the given integral is divergent.

Answer 6:

a)

The area enclosed by curve $y=6-x^2$ and $y=2-3x$ is shown below:



To find the intersection point(s), set one equation equal to other

$$2-3x = 6-x^2$$

$$x^2-3x-4=0$$

$$(x+1)(x-4)=0$$

$$x=-1,4$$

Hence the area is

$$A=\int_a^b\left[(\text{Upper Curve})-(\text{Lower Curve})\right]dx$$

$$=\int_{-1}^4\left[\left(6-x^2\right)-(2-3x)\right]dx$$

$$=\int_{-1}^4\left(4+3x-x^2\right)dx$$

$$=\left[4x+\frac{3}{2}x^2-\frac{x^3}{3}\right]_{-1}^4$$

$$=\left(16+24-\frac{64}{3}\right)-\left(-4+\frac{3}{2}+\frac{1}{3}\right)$$

$$=\boxed{\frac{125}{6}}$$

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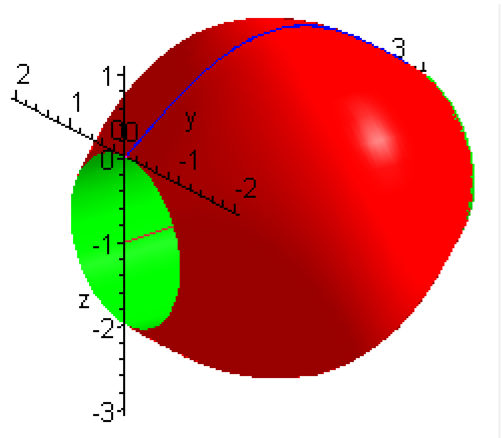
b)

Consider the volume after rotation:



The volume using washer method is

$$V=\int_0^{\pi}\pi\left[(\sin x+1)^2-1\right]dx$$
$$=\pi\int_0^{\pi}\left(\sin^2 x+2\sin x\right)dx$$
$$=\pi\int_0^{\pi}\left(\frac{1}{2}-\frac{1}{2}\cos 2x+2\sin x\right)dx$$
$$=\pi\left[\frac{1}{2}x-\frac{1}{4}\sin 2x-2\cos x\right]_0^{\pi}$$
$$=\pi\left(\frac{\pi}{2}+2+2\right)$$
$$=\boxed{\frac{\pi^2}{2}+4\pi}$$

c)

The average value of a function $f(x)$ over the interval $[a,b]$ is given by

$$f_{avg}=\frac{1}{b-a}\int_a^b f(x)dx$$

So,

$$f_{avg}=\frac{1}{3-0}\int_0^3\frac{x}{\sqrt{16+x^2}}dx$$
$$=\frac{1}{3}\int_0^3\frac{x}{\sqrt{16+x^2}}dx$$

Let $16+x^2=u$, then $2xdx=du\Rightarrow xdx=\frac{1}{2}du$

At $x=0, u=16$
At $x=3, u=25$

Thus,

$$f_{avg}=\frac{1}{3}\cdot\frac{1}{2}\int_{16}^{25}\frac{du}{\sqrt{u}}$$
$$=\frac{1}{6}\left[2\sqrt{u}\right]_{16}^{25}$$
$$=\frac{1}{3}\left[\sqrt{25}-\sqrt{16}\right]$$
$$=\frac{1}{3}(5-4)$$
$$=\boxed{\frac{1}{3}}$$

Answer 7:

a)

Consider the sequence $a_n=\frac{(3^n+1)^2}{6^n}$:

$$\lim_{n\to\infty}a_n=\lim_{n\to\infty}\frac{(3^n+1)^2}{6^n}$$
$$=\lim_{n\to\infty}\frac{3^{2n}+1+2\cdot 3^n}{6^n}$$
$$=\lim_{n\to\infty}\left(\frac{3^{2n}}{6^n}+\frac{1}{6^n}+2\cdot\frac{3^n}{6^n}\right)$$
$$=\lim_{n\to\infty}\left(\left(\frac{3}{2}\right)^n+\left(\frac{1}{6}\right)^n+2\cdot\left(\frac{1}{2}\right)^n\right)$$
$$=\infty+0+2\cdot 0 \qquad \left[\lim_{n\to\infty}a^n=\begin{cases}\infty, a>1\\ 0, 0<a<1\end{cases}\right]$$
$$=\infty$$

Hence the value of a_n is not finite, therefore $\boxed{\text{limit does not exist}}$.

b)

Consider the sequence:

$$a_n=\ln(1+2n^2)-\ln(30+2n^2)=\ln\left(\frac{1+2n^2}{30+2n^2}\right)$$

Find the limit of sequence as

$$\lim_{n\to\infty}a_n=\lim_{n\to\infty}\ln\left(\frac{1+2n^2}{30+2n^2}\right)$$
$$=\lim_{n\to\infty}\ln\left(\frac{\frac{1}{n^2}+2}{\frac{30}{n^2}+2}\right)$$
$$=\ln\left(\frac{0+2}{0+2}\right)$$
$$=\ln 1$$
$$=0$$

Thus, the limit of the sequence is $\boxed{0}$.

Answer 8:

a)

Consider the series:

$$\sum_{n=2}^{\infty} a_n = \sum_{n=2}^{\infty} \frac{(-1)^n \sqrt{1+n^3}}{n^2}$$

Here $\frac{\sqrt{1+n^3}}{n^2}$ is positive and monotone decreasing and

$$\lim_{n\to\infty} \frac{\sqrt{1+n^3}}{n^2} = \lim_{n\to\infty} \frac{\sqrt{\frac{1}{n^3}+1}}{n^{\frac{1}{2}}}$$
$$= 0$$

Thus, by alternating series test series converges.

Now check absolute convergence $|a_n| = \frac{\sqrt{1+n^3}}{n^2}$. Let $b_n = \frac{1}{n^{\frac{1}{2}}}$

Since $b_n = \frac{1}{n^{\frac{1}{2}}}$ is a divergent series by p-series test and

$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty} \frac{\frac{\sqrt{1+n^3}}{n^2}}{\frac{1}{n^{\frac{1}{2}}}}$$
$$= \lim_{n\to\infty} \frac{n^{\frac{1}{2}} \cdot n^{\frac{3}{2}} \sqrt{\frac{1}{n^3}+1}}{n^2}$$
$$= \lim_{n\to\infty} \frac{n^2 \sqrt{\frac{1}{n^3}+1}}{n^2}$$
$$= \lim_{n\to\infty} \sqrt{\frac{1}{n^3}+1}$$
$$= \sqrt{0+1}$$
$$= 1$$

So, by limit comparison test series diverges.

Thus, limit converges conditionally.

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b)

Consider the series:

$$\sum_{n=2}^{\infty} a_n = \sum_{n=2}^{\infty} \frac{(-3)^{3n}}{5+e^n}$$
$$= \sum_{n=2}^{\infty} \frac{(-1)^n (27)^n}{5+e^n}$$

Check the absolute convergence $|a_n| = \left|\frac{(-1)^n (27)^n}{5+e^n}\right| = \frac{(27)^n}{5+e^n}$, by divergent test as

$$\lim_{n\to\infty} \frac{27^n}{5+e^n} = \lim_{n\to\infty} \frac{1}{\frac{5}{27^n} + \left(\frac{e}{27}\right)^n}$$
$$= \infty$$
$$\neq 0$$

Thus, by divergent test, series diverges.

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c)

Consider the series:

$$\sum_{n=2}^{\infty} a_n = \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$$

We will use function

$$f(x) = \frac{1}{x(\ln x)^2}$$

This function is clearly positive and if we make x larger the denominator will get larger and larger and hence function is also decreasing.

Therefore all we need to do is determine the convergence of following integral

$$\int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx = \lim_{t\to\infty} \int_2^{t} \frac{1}{x(\ln x)^2}\,dx$$
$$= \lim_{t\to\infty} \left[-\frac{1}{\ln x}\right]_2^t \qquad [u = \ln x]$$
$$= \lim_{t\to\infty} \left(\frac{1}{\ln 2} - \frac{1}{\ln t}\right)$$
$$= \frac{1}{\ln 2} - 0 = \frac{1}{\ln 2}$$

The integral is convergent, so the series must be convergent by the integral test.

The series is always positive, so the series is absolutely convergent.

Answer 9:

Consider the series $\sum_{n=0}^{\infty} \frac{(x+1)^n}{(n+1)2^n}$:

Find the radius of convergence as

$$\begin{aligned}
r &= \lim_{n\to\infty} \frac{a_n}{a_{n+1}} \\
&= \lim_{n\to\infty} \frac{\frac{1}{(n+1)2^n}}{\frac{1}{(n+2)2^{n+1}}} \\
&= \lim_{n\to\infty} \frac{2(n+2)}{(n+1)} \\
&= \lim_{n\to\infty} \frac{2\left(1+\frac{2}{n}\right)}{\left(1+\frac{1}{n}\right)} \\
&= \frac{2(1+0)}{(1+0)} \\
&= 2
\end{aligned}$$

Since the center is at $x=-1$, the interval of convergence is

$$\begin{aligned}
-1-2 < x < -1+2 \\
-3 < x < 1
\end{aligned}$$

Second method:

By series ratio test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(x+1)^{n+1}}{(n+2)2^{n+1}}}{\frac{(x+1)^n}{(n+1)2^n}}\right| = \lim_{n\to\infty}\left|\frac{(x+1)(n+1)}{2(n+2)}\right| = \frac{1}{2}|x+1|\lim_{n\to\infty}\left|\frac{1+\frac{1}{n}}{1+\frac{2}{n}}\right| = \frac{1}{2}|x+1|\left|\frac{1+0}{1+0}\right|$$

$$= \frac{1}{2}|x+1|$$

The series converges, when $\frac{1}{2}|x+1| < 1$

$$\begin{aligned}
|x+1| &< 1 \\
|x+1| &< 2 \\
-2 < x+1 &< 2 \\
-3 < x &< 1
\end{aligned}$$

Now check at endpoint $x=-3$:

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(-3+1)^n}{(n+1)2^n} &= \sum_{n=0}^{\infty} \frac{(-2)^n}{(n+1)2^n} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{(n+1)2^n} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)}
\end{aligned}$$

Here $a_n = \frac{1}{n+1}$ is positive and decreasing function and

$$\lim_{n\to\infty} \frac{1}{n+1} = 0$$

By alternating series test, series converges at $x=-3$.

Now check at another endpoint $x=1$:

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(1+1)^n}{(n+1)2^n} &= \sum_{n=0}^{\infty} \frac{2^n}{(n+1)2^n} \\
&= \sum_{n=0}^{\infty} \frac{1}{(n+1)}
\end{aligned}$$

Let $a_n = \frac{1}{n+1}$ and $b_n = \frac{1}{n}$

Since $b_n = \frac{1}{n}$ diverges and

$$\lim_{n\to\infty}\left(\frac{\frac{1}{n+1}}{\frac{1}{n}}\right) = 1$$

By the limit comparison test series diverges at $x=1$.

Hence radius of convergence is $-3 \le x < 1$

Answer 10:

a)

We already know the well known power series of $\ln(1+x)$ is

$$\ln(1+x)=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$$

Replace x by $2x$ to obtain

$$\begin{aligned}\ln(1+2x)&=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(2x)^n}{n}\\&=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2^n\cdot x^n}{n}\end{aligned}$$

Multiply both sides by x^2 to obtain

$$\begin{aligned}x^2\ln(1+2x)&=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2^n\cdot x^n\cdot x^2}{n}\\&=\boxed{\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2^n}{n}x^{n+2}}\end{aligned}$$

Find the radius of convergence as

$$\begin{aligned}r&=\lim_{n\to\infty}\frac{a_n}{a_{n+1}}\\&=\lim_{n\to\infty}\frac{\frac{2^n}{n}}{\frac{2^{n+1}}{n+1}}\\&=\lim_{n\to\infty}\frac{n+1}{2n}\\&=\lim_{n\to\infty}\frac{1}{2}\left(1+\frac{1}{n}\right)\\&=\frac{1}{2}(1+0)\\&=\frac{1}{2}\end{aligned}$$

Since the center is at $x=0$, the interval of convergence is

$$-\frac{1}{2}<x<\frac{1}{2}$$

Now check at endpoint $x=-\frac{1}{2}$:

$$\begin{aligned}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2^n}{n}x^{n+2}&=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2^n}{n}\left(\frac{1}{2}\right)^{n+2}\\&=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2^n}{n}\cdot\left(\frac{1}{2^n\cdot 2^2}\right)\\&=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{4n}\end{aligned}$$

Here $a_n=\frac{1}{4n}$ is positive and decreasing function and

$$\lim_{n\to\infty}\frac{1}{4n}=0$$

By alternating series test, series converges at $x=-\frac{1}{2}$.

Now check at another endpoint $x=\frac{1}{2}$:

$$\begin{aligned}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2^n}{n}x^{n+2}&=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2^n}{n}\left(-\frac{1}{2}\right)^{n+2}\\&=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2^n}{n}\cdot(-1)^{n+2}\left(\frac{1}{2^n\cdot 2^2}\right)\\&=\sum_{n=1}^{\infty}(-1)^{2n+1}\frac{1}{4n}\\&=\sum_{n=1}^{\infty}\frac{-1}{4n}\end{aligned}$$

By the harmonic series, diverges at $x=-\frac{1}{2}$.

Hence radius of convergence is $\boxed{-\frac{1}{2}<x\le\frac{1}{2}}$

b)

Consider the given series

$$S(x)=\sum_{n=1}^{\infty}nx^{2n-1}$$

Integrating both sides,

$$\int S(x)\,dx = \sum_{n=1}^{\infty} n \int x^{2n-1}$$
$$= \sum_{n=1}^{\infty} n\left(\frac{x^{2n}}{2n}\right)$$
$$= \frac{1}{2}\sum_{n=1}^{\infty}\left(x^2\right)^n$$
$$= \frac{1}{2}\cdot\frac{x^2}{1-x^2} \qquad \left[S_\infty = \frac{a}{1-r}, a = x^2, r = x^2\right]$$

Hence

$$S(x) = \frac{1}{2}\cdot\frac{d}{dx}\left(\frac{x^2}{1-x^2}\right)$$
$$= \frac{1}{2}\cdot\frac{2x}{\left(1-x^2\right)^2}$$
$$= \boxed{\frac{x}{\left(1-x^2\right)^2}}$$

The series converges for $\left|x^2\right| < 1 \Rightarrow \boxed{-1 < x < 1}$

Bonus question:

Consider the integral

$$\int_0^{\frac{\pi}{2}} f(\cos x)\,dx$$

Let $u = \frac{\pi}{2} - x$ then $du = -dx \Rightarrow dx = -du$

When $x = 0$ then $u = \frac{\pi}{2}$

When $x = \frac{\pi}{2}$ then $u = 0$

Hence the integral becomes

$$\int_0^{\frac{\pi}{2}} f(\cos x)\,dx = -\int_{\frac{\pi}{2}}^{0} f\left(\cos\left(\frac{\pi}{2} - u\right)\right)du$$
$$= -\int_{\frac{\pi}{2}}^{0} f(\sin u)\,du$$
$$= \int_0^{\frac{\pi}{2}} f(\sin u)\,du \qquad \left[-\int_b^a f(x)\,dx = \int_a^b f(x)\,dx\right]$$
$$= \int_0^{\frac{\pi}{2}} f(\sin x)\,dx \qquad \left[\int_a^b f(t)\,dt = \int_a^b f(x)\,dx\right]$$

Therefore $\boxed{\int_0^{\frac{\pi}{2}} f(\cos x)\,dx = \int_0^{\frac{\pi}{2}} f(\sin x)\,dx}$