CARLETON UNIVERSITY

FINAL EXAMINATION
December 2015

DURATION: 3 HOURS

Department Name and Course Number: Mathematics and Statistics, MATH 2007A,B
Course Instructor(s): Dr. E. Hua, Dr. S. Melkonian

AUTHORIZED MEMORANDA
Non-programmable, non-graphic calculators

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[Marks]

[5] 1. $\lim_{x\to 1} \dfrac{2x^3 - 3x^2 + 1}{x^3 - x^2 - x + 1} =$

(a) 2 (b) $\dfrac{3}{2}$ (c) 0 (d) ∞ (e) None of these

[5] 2. $\int 4x(x^2+3)^8\,dx =$

(a) $\dfrac{1}{18}(x^2+3)^9 + C$ (b) $\dfrac{2}{9}(x^2+3)^9 + C$ (c) $\dfrac{4}{9}x(x^2+3)^9 + C$ (d) $\dfrac{4}{9}(x^2+3)^9 + C$

(e) None of these

[5] 3. $\int_0^1 xe^{-x}\,dx =$

(a) $1 - \dfrac{2}{e}$ (b) -1 (c) $2e^{-1} - 1$ (d) $2e^{-1} + 1$ (e) None of these

[5] 4. $\int_0^2 \sqrt{4 - x^2}\,dx =$ (a) $\dfrac{\pi}{2}$ (b) $\dfrac{\pi}{4}$ (c) π (d) 0 (e) None of these

[5] 5. The partial fraction decomposition of $\dfrac{x^2+1}{(x+1)^3(x^2+9)^2}$ has the form

(a) $\dfrac{A}{x+1} + \dfrac{B}{x^2+9}$

(b) $\dfrac{A}{x+1} + \dfrac{B}{(x+1)^2} + \dfrac{C}{(x+1)^3} + \dfrac{Dx+E}{x^2+9} + \dfrac{Fx+G}{(x^2+9)^2}$

(c) $\dfrac{A}{x+1} + \dfrac{Bx+C}{x^2+9}$

(d) $\dfrac{A}{x+1} + \dfrac{B}{(x+1)^2} + \dfrac{C}{(x+1)^3} + \dfrac{D}{x^2+9} + \dfrac{E}{(x^2+9)^2}$

(e) None of these

[5] 6. The improper integral $\int_0^1 \frac{1}{x}\,dx$

(a) Diverges (b) Converges to -1 (c) Converges to 1
(d) Converges to 0 (e) None of these

[5] 7. The polar coordinates of $(x,y) = (\sqrt{3}, -1)$ are $(r,\theta) =$

(a) $\left(2, \frac{2\pi}{3}\right)$ (b) $\left(2, \frac{5\pi}{6}\right)$ (c) $\left(2, \frac{7\pi}{6}\right)$ (d) $\left(2, -\frac{\pi}{6}\right)$ (e) None of these

[5] 8. The slope of the tangent to the polar curve $r = \sin(\theta)$ at $\theta = \frac{\pi}{3}$ is

(a) $-\sqrt{3}$ (b) $\sqrt{3}$ (c) $\frac{1}{\sqrt{3}}$ (d) $-\frac{1}{\sqrt{3}}$ (e) None of these

[5] 9. The sum of the series $\sum_{n=1}^{\infty} 2^{3-2n} 3^n$ is

(a) 32 (b) 3 (c) 24 (d) Diverges (e) None of these

[5] 10. The length of the parametric curve $x = \cos(2t)$, $y = \sin(2t)$, $0 \le t \le 2\pi$, is

(a) π (b) 2π (c) 4π (d) $2\sqrt{2}\,\pi$ (e) None of these

[5] 11. The Taylor series of $f(x) = e^x$ about $a = 2$ is

(a) $\sum_{n=0}^{\infty} \frac{1}{n!}(x-2)^n$ (b) $\sum_{n=0}^{\infty} \frac{1}{n!}(x+2)^n$ (c) $\sum_{n=0}^{\infty} \frac{e^2}{n!}(x+2)^n$

(d) $\sum_{n=0}^{\infty} \frac{e^2}{n!}(x-2)^n$ (e) None of these

[5] 12. The coefficient of x^3 in the Maclaurin series (binomial series) of $f(x) = \sqrt{1+x}$ is

(a) $\frac{1}{8}$ (b) $\frac{1}{16}$ (c) $-\frac{1}{8}$ (d) $-\frac{1}{16}$ (e) None of these

[10] 13. Evaluate the following integrals:

(a) $\int x^2(x^3+1)^4\,dx$ (b) $\int 4\sin^3(x)\cos(x)\,dx$

[10] 14. Determine whether the series converges absolutely, converges conditionally, or diverges. Justify your answer.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{\sqrt{n^4+1}}$ (b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{3^n+1}$

[10] 15. Consider the parametric curve $x = t^3$, $y = \frac{1}{2}t^2 - 2t + 1$.

(a) Find $\frac{dy}{dx}$.

(b) Determine the values of t and the corresponding points (x, y) at which the tangent is horizontal or vertical.

[10] 16. Consider the power series $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n}(x-1)^n$.

(a) Find the radius of convergence.

(b) Find the interval of convergence.

Answers and Solutions

1.(b) 2.(b) 3.(a) 4.(c) 5.(b) 6.(a) 7.(d) 8.(a) 9.(c) 10.(c) 11.(d) 12.(b)

13. (a) $\int x^2(x^3+1)^4\,dx = \frac{1}{3}\int u^4\,du = \frac{1}{15}u^5 + C = \frac{1}{15}(x^3+1)^5 + C$, $u = x^3+1$.

    (b) $\int 4\sin^3(x)\cos(x)\,dx = \int 4u^3\,du = u^4 + C = \sin^4(x) + C$, $u = \sin(x)$.

14. (a) $\sum_{n=1}^{\infty} \frac{n^2}{\sqrt{n^4+1}}$ diverges by the nth-term test, since $\lim_{n\to\infty} \frac{n^2}{\sqrt{n^4+1}} = 1 \neq 0$.

    (b) $\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{3^n+1}$ converges absolutely by the comparison test, because $\sum_{n=0}^{\infty} \left|\frac{(-1)^n 2^n}{3^n+1}\right| = \sum_{n=0}^{\infty} \frac{2^n}{3^n+1}$ and $\frac{2^n}{3^n+1} \leq \left(\frac{2}{3}\right)^n$, and $\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n$ converges since it is geometric with $|r| < 1$.
    Alternatively, by the ratio test,
    $\left|\frac{a_{n+1}}{a_n}\right| = \frac{2^{n+1}}{3^{n+1}+1}\frac{3^n+1}{2^n} = \frac{2}{3}\frac{1+3^{-n}}{1+3^{-(n+1)}} \to \frac{2}{3} < 1$ as $n \to \infty$.

15. (a) $\frac{dx}{dt} = 3t^2$, $\frac{dy}{dt} = t-2 \;\Rightarrow\; \frac{dy}{dx} = \frac{t-2}{3t^2}$.
    (b) The tangent is horizontal at $t = 2$, i.e., $x = 8$ and $y = -1$, and it is vertical at $t = 0$, i.e., $x = 0$ and $y = 1$.

16. (a) $R = \lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{2^{n+1}}{2^n}\right| = \lim_{n\to\infty} 2 = 2$.

    (b) $|x-1| < 2 \;\Rightarrow\; -2 < x-1 < 2 \;\Rightarrow\; -1 < x < 3$. $x = -1 \;\Rightarrow\; \sum_{n=0}^{\infty}(-1)^n$ diverges (by the nth-term test, since $\lim_{n\to\infty}(-1)^n \neq 0$), and $x = 3 \;\Rightarrow\; \sum_{n=0}^{\infty} 1$ diverges (by the nth-term test, since $\lim_{n\to\infty} 1 \neq 0$). Hence $I = (-1, 3)$.