University of Ottawa

Department of Mathematics and Statistics

MAT 1302A: Mathematical Methods II
Professor: Rostislav Devyatov

First Midterm Exam – Version A

February 3, 2017

Surname ____________________ First Name ____________________

Student # ____________________ DGD ____________________

Instructions:

(a) You have 80 minutes to complete this exam.
(b) All work to be considered for grading should be written in the space provided. The reverse side of pages is for scrap work. If you find that you need extra space in order to answer a particular question, you should continue on the reverse side of the page and indicate this **clearly**. Otherwise, the work written on the reverse side of pages will not be considered for marks.
(c) Write your student number at the top of each page in the space provided.
(d) No notes, books, scrap paper, calculators or other electronic devices are allowed.
(e) You are strongly recommended to write in **pen**, not pencil.
(f) You may use the last page of the exam as scrap paper.
(g) Cellular phones, unauthorized electronic devices or course notes (unless an open-book exam) are not allowed during this exam. **Phones and devices must be turned off and put away in your bag**. Do not keep them in your possession, such as in your pockets. If caught with such a device or document, the following may occur: you will be asked to leave immediately the exam and academic fraud allegations will be filed which may result in you obtaining a 0 (zero) for the exam.

By signing below, you acknowledge that you have ensured that you are complying with the above statement.

Signature ____________________

Please do not write in the table below.

Question	1	2	3	4	5	6	Total
Maximum	4	3	2	4	5	3	21
Grade							

1. [**4 points**] Determine if the matrix equation

$$\begin{bmatrix} 1 & -2 & 2 & -1 \\ -1 & 2 & 0 & -1 \\ 2 & -4 & 4 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} -2 \\ -4 \\ -4 \end{bmatrix}$$

is consistent or inconsistent. If the linear system is consistent, write its general solution in vector parametric form.

Solution:

Reduced echelon form of the augmented matrix of the system is obtained as follows:

$$\left[\begin{array}{cccc|c} 1 & -2 & 2 & -1 & -2 \\ -1 & 2 & 0 & -1 & -4 \\ 2 & -4 & 4 & -2 & -4 \end{array}\right] \xrightarrow{R_1+R_2\to R_2} \left[\begin{array}{cccc|c} 1 & -2 & 2 & -1 & -2 \\ 0 & 0 & 2 & -2 & -6 \\ 2 & -4 & 4 & -2 & -4 \end{array}\right]$$

$$\xrightarrow{-2R_1+R_3\to R_3} \left[\begin{array}{cccc|c} 1 & -2 & 2 & -1 & -2 \\ 0 & 0 & 2 & -2 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \text{ E.F.}$$

$$\xrightarrow{\frac{1}{2}R_2\to R_2} \left[\begin{array}{cccc|c} 1 & -2 & 2 & -1 & -2 \\ 0 & 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-2R_2+R_1\to R_1} \left[\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 4 \\ 0 & 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \text{ R.R.E.F.}$$

From the E.F. or the R.R.E.F. it is clear that the system is consistent. The pivot columns are the first and the third columns, so x and z are basic, and y and t are free. The R.R.E.F. is the augmented matrix of the linear system

$$\begin{array}{ccccc} x & -2y & & +t & = 4 \\ & & z & -t & = -3 \end{array}$$

Therefore the general solution is

$$\begin{cases} x = 2y - t + 4 \\ y : \text{free} \\ z = t - 3 \\ t : \text{free} \end{cases}$$

and in vector parametric form:

$$\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 2y - t + 4 \\ y \\ t - 3 \\ t \end{bmatrix} = y \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ 0 \\ -3 \\ 0 \end{bmatrix}$$

Grading Scheme:

- 1 pt for writing the correct augmented matrix.
- 2 pts for the row reduction to obtain the R.E.F.
- 1 pt for writing the general solution in vector parametric form. Then do not get this point if they write the general solution in another form.
- See next page for more criteria!

- If row reduction contains computational mistakes, but the resulting matrix is in RREF:
  - Deduct 0.75pt (from 2pt) for each computatinal mistake
  - If the final answer in vector parametric form is consistent with the result of row reduction, give 1pt for the final answer
- If row reduction ends with a matrix in EF, but not in RREF:
  - If the general solution is present, not necessarily in vector parametric form, but each basic variable is expressed **in terms of free variables only** (nothing like $x = 2y - 2z+$ something, $z = t - 3$), this part (from the beginning of row reduction until this form of general solution) gives 2pt. The final answer in vector parametric form gives 1pt more. computational mistakes are treated as in the previous criterion.
  - If there is no such general solution, the row reduction gives 1pt maximum, -0.5pt for each computational mistake.
- If row reduction ends with a matrix, which is not even in EF, the row reduction gives 0.5pt if there are no mistakes.
- If the augmented matrix is written incorrectly:
  - If the resulting SLE (with the wrong augmented matrix) is not much easier than the original one, they don't get the first 1pt for the augmented matrix, but can receive up to 2pt for row reduction of this matrix and up to 1pt for the solution of this wrong SLE in vector parametric form.
  - If the new SLE is much easier than the old one (for example, all constant terms became 0), they don't get the first 1pt for the augmented matrix, and the points for the subsequent steps are divided by 2.

2. [**3 points**] Determine all values of the parameter h such that the linear system

$$\begin{cases} 2x_1 &= 2x_2 + x_3 + 4 \\ x_1 &= x_2 - hx_3 - 3 \end{cases}$$

is inconsistent. You should justify your answer.

Solution: The linear system can be written in the standard form as

$$\begin{cases} 2x_1 - 2x_2 - x_3 &= 4 \\ x_1 - x_2 + hx_3 &= -3 \end{cases}$$

The augmented matrix of this system can be reduced to an E.F. as follows

$$\left[\begin{array}{ccc|c} 2 & -2 & -1 & 4 \\ 1 & -1 & h & -3 \end{array}\right] \xrightarrow{-\frac{1}{2}R_1+R_2\to R_2} \left[\begin{array}{ccc|c} 2 & -2 & -1 & 4 \\ 0 & 0 & h+\frac{1}{2} & -5 \end{array}\right]$$

The system is inconsistent if and only if $h = -\frac{1}{2}$.

Grading Scheme:

- 1 pt for writing the correct augmented matrix.
- 1 pt for the correct row reduction.
- 1 pt for the final answer that is consistent with the result of the row reduction, even if their final answer is incorrect.
- If the initial augmented matrix is incorrect but contains three variables, it is still possible to get 1 pt for the correct row reduction and 1 pt for the final answer.
- If the initial augmented matrix is incorrect and contains less variables, the maximums are 0.5pt for the correct row reduction and 0.5pt for the final answer.

3. Compute the following:

(a) [**1 point**] $\begin{bmatrix} 4 & -3 & -1 \\ 2 & 0 & -1 \\ -4 & 1 & 3 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$

Solution: $\begin{bmatrix} -12 \\ -5 \\ 12 \end{bmatrix}$

(b) [**1 point**] $-3\mathbf{u} + 2\mathbf{v}$ where $\mathbf{u} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$.

Solution: $\begin{bmatrix} 3 \\ 4 \\ -7 \end{bmatrix}$

Grading Scheme: In each part, take off .5 points for each incorrect entry.

4. [**4 points**] Set

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \\ -1 \end{bmatrix}, \ \mathbf{a}_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 3 \end{bmatrix}, \ \mathbf{a}_3 = \begin{bmatrix} -1 \\ 0 \\ -1 \\ 2 \end{bmatrix}, \ \text{and } \mathbf{b} = \begin{bmatrix} -3 \\ 1 \\ -4 \\ 2 \end{bmatrix}.$$

Does $\mathbf{b}$ belong to Span$\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$? You should justify your answer.

Solution: We should reduce the following matrix to an E.F.:

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & -3 \\ -2 & 1 & 0 & 1 \\ 3 & 1 & -1 & -4 \\ -1 & 3 & 2 & 2 \end{array}\right] \xrightarrow{2R_1+R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 2 & -1 & -3 \\ 0 & 5 & -2 & -5 \\ 3 & 1 & -1 & -4 \\ -1 & 3 & 2 & 2 \end{array}\right] \xrightarrow{-3R_1+R_3 \to R_3}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & -3 \\ 0 & 5 & -2 & -5 \\ 0 & -5 & 2 & 5 \\ -1 & 3 & 2 & 2 \end{array}\right] \xrightarrow{R_1+R_4 \to R_4} \left[\begin{array}{ccc|c} 1 & 2 & -1 & -3 \\ 0 & 5 & -2 & -5 \\ 0 & -5 & 2 & 5 \\ 0 & 5 & 1 & -1 \end{array}\right] \xrightarrow{R_2+R_3 \to R_3}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & -3 \\ 0 & 5 & -2 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 5 & 1 & -1 \end{array}\right] \xrightarrow{-R_2+R_4 \to R_4} \left[\begin{array}{ccc|c} 1 & 2 & -1 & -3 \\ 0 & 5 & -2 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 4 \end{array}\right] \xrightarrow{R_3 \leftrightarrow R_4} \left[\begin{array}{ccc|c} 1 & 2 & -1 & -3 \\ 0 & 5 & -2 & -5 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Since the last column is not a pivot column, it follows that $\mathbf{b}$ is a linear combination of $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$.

Grading Scheme:

- 1 pt for the correct matrix.
- 2 pts for the correct row reduction.
- 1 pt for the final answer which is consistent with the result of their row reduction, even if the final answer is incorrect.
- If row reduction contains computational mistakes, but the resulting matrix is in EF, deduct 0.5pt (from 2pt) for each computatinal mistake
- If row reduction ends with a matrix, which is "almost in EF" (it is already clear whether the system is consistent or no, for example, the last operation $R_3 \leftrightarrow R_4$ is missing in the solution above):
  - Max 1.5pt for such row reduction
  - 1pt if the final answer is consistent with this "almost EF"
- If row reduction ends with a matrix, which is really not in EF, the row reduction gives 1pt if there are no mistakes. Final answer does not give any points.
- If the augmented matrix is written incorrectly:
  - If the resulting SLE (with the wrong augmented matrix) is not much easier than the original one, they don't get the first 1pt for the augmented matrix, but can receive up to 2pt for row reduction of this matrix and up to 1pt for the solution of this wrong SLE in vector parametric form.
  - If the new SLE is much easier than the old one (for example, all constant terms became 0), they don't get the first 1pt for the augmented matrix, and the points for the subsequent steps are divided by 2.

5. [**5 points**] For each of the following statements, indicate if it is true or false. You will receive 1 point for every correct answer and lose .5 points for every incorrect answer (but you cannot receive a negative mark on this question).

(a) A homogeneous linear system with n equations and n variables always has a unique solution.

Solution: False.

(b) For any vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ in $\mathbb{R}^3$, the vector $-2\mathbf{a}+\mathbf{b}-3\mathbf{c}$ always belongs to Span$\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$.

Solution: True.

(c) A consistent linear system with a 4×6 augmented matrix will have infinitely many solutions.

Solution: True.

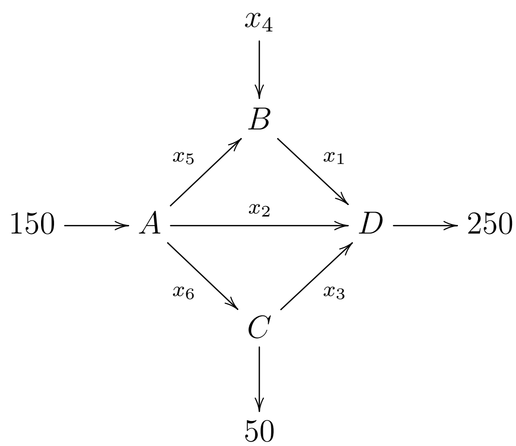
(d) If a 4×5 matrix A has 4 pivot positions, then for every vector $\mathbf{b}$ in $\mathbb{R}^4$ the matrix equation $A\mathbf{x} = \mathbf{b}$ has a solution.

Solution: True.

(e) Two row equivalent matrices which are both in echelon form must be equal.

Solution: False.

6. Consider the traffic flow described by the following diagram. The letters A through D label intersections. The arrows indicate the direction of flow (all roads are one-way) and their labels indicate flow in cars per minute.



(a) **[2 points]** Write down a linear system describing the traffic flow, i.e., all constraints on the variables $x_i, i = 1, \ldots, 6$. (You do not need to solve the linear system.)

Solution:

$$\begin{cases} A: & 150 = x_2 + x_5 + x_6 \\ B: & x_4 + x_5 = x_1 \\ C: & x_6 = x_3 + 50 \\ D: & x_1 + x_2 + x_3 = 250 \\ \text{total}: & x_4 + 150 = 250 + 50 \end{cases}$$

Grading Scheme: -.5 pts for each missing or incorrect equation.

(b) **[1 point]** The reduced row echelon form of the augmented matrix corresponding to the linear system in part (a) is:

$$\left[\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & -1 & 0 & 150 \\ 0 & 1 & 0 & 0 & 1 & 1 & 150 \\ 0 & 0 & 1 & 0 & 0 & -1 & -50 \\ 0 & 0 & 0 & 1 & 0 & 0 & 150 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Write down the general flow pattern.

Solution: The general flow pattern is the general solution of the linear system, given by

$$\begin{cases} x_1 & = x_5 + 150 \\ x_2 & = -x_5 - x_6 + 150 \\ x_3 & = x_6 - 50 \\ x_4 & = 150 \\ x_5 & : \text{free} \\ x_6 & : \text{free} \end{cases}$$

Grading Scheme: All or nothing.