Mast 218 Final December 2012 Solutions ①

1. $x = t^3 - 3t \qquad y = t^3 - 3t^2$

(a) $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{3t^2-6t}{3t^2-3} = \dfrac{t^2-2t}{t^2-1}$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt} = \frac{(2t-2)(t^2-1)-(t^2-2t)\cdot 2t}{3(t^2-1)(t^2-1)^2} =$$

$$= \frac{\cancel{2t^3}-2t^2-2t+2-\cancel{2t^3}+4t^2}{3(t^2-1)^3} = \frac{2t^2-2t+2}{3(t^2-1)^3}$$

(b) tangent is horizontal if $\dfrac{dy}{dx}=0$ or

$\dfrac{dy}{dt}=0 \Leftrightarrow t(t-2)=0 \Rightarrow t=0$ or $t=2$

Hence for $x=0$ & $y=0$ point: $(0,0)$

or for $x=2$ & $y=-4$ point: $(2,-4)$

tangent is vertical if $\dfrac{dx}{dt}=0$ so

$3t^2-3=0 \Rightarrow t^2=1 \Rightarrow t=\pm 1$

Hence for $x=-2$ & $y=-2$ point: $(-2,-2)$

or for: $x=2$ & $y=-4$ point: $(2,-4)$

2. ⓐ $\gamma_2 : r = 3\sin\theta \quad \Big| \cdot r$

$r^2 = 3r\sin\theta \qquad 0 \le \theta \le \pi$

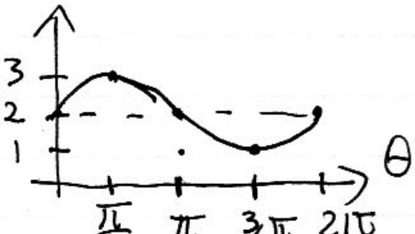
But $x^2+y^2 = r^2$ & $y = r\sin\theta$

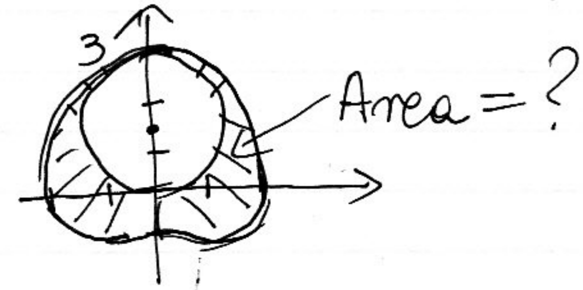
So: $x^2+y^2 = 3y \iff x^2+y^2-3y=0$

$x^2+y^2-3y+\left(\frac{3}{2}\right)^2-\left(\frac{3}{2}\right)^2=0$

$x^2+\left(y-3/2\right)^2=\left(\frac{3}{2}\right)^2$ a circle with the center at $(0, 3/2)$ & the radius of $3/2$

γ_1:





ⓑ Area inside γ_1 − area of the circle

Area inside $\gamma_1 = \frac{1}{2}\int_0^{2\pi}(2+\sin\theta)^2\,d\theta =$

$= \frac{1}{2}\int_0^{2\pi}(4+4\sin\theta+\sin^2\theta)\,d\theta \quad \overline{\sin^2\theta = \frac{1-\cos 2\theta}{2}}$

$= \frac{1}{2}\int_0^{2\pi}\left(4+4\sin\theta+\frac{1}{2}-\frac{1}{2}\cos 2\theta\right)d\theta =$

$= \frac{1}{2}\left[\frac{9}{\cancel{2}}\cdot\cancel{2}\pi - 4\cos\theta\Big|_0^{2\pi} - \frac{1}{4}\sin 2\theta\Big|_0^{2\pi}\right] = \frac{9}{2}\pi$

So Area between $= \frac{9}{2}\pi - \left(\frac{3}{2}\right)^2\cdot\pi = \frac{9\cdot 2-9}{4}\pi = \underline{\underline{\frac{9}{4}\pi}}$

3. $r(t) = \langle 2t^{3/2}, \cos 2t, \sin 2t \rangle$

$$L = \int_1^3 |r'(t)|\, dt$$

$$r'(t) = \langle \cancel{2} \cdot \frac{3}{\cancel{2}} t^{1/2}, -2\sin 2t, 2\cos 2t \rangle$$

$$|r'(t)| = \sqrt{9t + 4\sin^2 2t + 4\cos^2 2t} = \sqrt{9t+4}$$

Hence

$$L = \int_1^3 \sqrt{9t+4}\, dt \;\underset{du = 9dt}{\overset{u = 9t+4}{=}}\; \frac{1}{9}\int_1^3 \sqrt{9t+4} \cdot 9dt$$

$t = 1 \Rightarrow u = 9+4 = 13$
$t = 3 \Rightarrow u = 27+4 = 31$

$$= \frac{1}{9}\int_{13}^{31} u^{1/2}\, du = \frac{1}{9}\, u^{3/2} \cdot 2/3 \Big|_{13}^{31} =$$

$$= \frac{2}{27}\left[31^{3/2} - 13^{3/2}\right] \simeq \underline{\underline{9,313}}$$

4. $\vec{AB} = \langle -3, -2, 9 \rangle$ both belong to PL

(a) $\vec{AC} = \langle -1, 2, -5 \rangle$ $\Rightarrow$ $n \perp PL$ is their cross product:

$$n = \vec{AB} \times \vec{AC} = \langle 10-18, -(15+9), -6-2 \rangle =$$
$$= \langle -8, -24, -8 \rangle$$

Any vector of the same direction is normal to PL so choose the parallel one

$$n^{new} = -\frac{1}{8} n = \langle 1, 3, 1 \rangle$$

PL: $(x-x_0) + 3(y-y_0) + 1(z-z_0) = 0$ $A \in PL$ so

$(x-2) + 3(y-1) + (z-1) = 0$ is the eq. of PL

or $\underline{x + 3y + z = 6}$,

(b) L is the line passing through $B = (-1, -1, 10)$ & parallel to $\langle 1, 3, 1 \rangle$

parametric equations: $\begin{cases} x = -1 + t \\ y = -1 + 3t \\ z = 10 + t \end{cases}$,

implicit equations:

$$\frac{x+1}{1} = \frac{y+1}{3} = \frac{z-10}{1}$$

5. $\sin(xyz) = x^2y^2 + z^2 - 1$ or

$x^2y^2 + z^2 - 1 - \sin(xyz) = 0$ (*)

(a) $\frac{\partial z}{\partial x}$ take $\frac{\partial}{\partial x}$ of (*)

$$2xy^2 + 2z\frac{\partial z}{\partial x} - \cos(xyz)\left[yz + xy\frac{\partial z}{\partial x}\right] = 0$$

$$\frac{\partial z}{\partial x}\left(2z - xy\cos(xyz)\right) = -2xy^2 + yz\cos(xyz)$$

$$\frac{\partial z}{\partial x} = \frac{yz\cos(xyz) - 2xy^2}{2z - xy\cos(xyz)}$$

$\frac{\partial z}{\partial y} = ?$ take $\frac{\partial}{\partial y}(*)$

$$2x^2y + 2z\frac{\partial z}{\partial y} - \cos(xyz)\left[xz + xy\frac{\partial z}{\partial y}\right] = 0$$

$$\frac{\partial z}{\partial y}\left[2z - xy\cos(xyz)\right] = xz\cos(xyz) - 2x^2y$$

$$\Rightarrow \frac{\partial z}{\partial y} = \frac{xz\cos(xyz) - 2x^2y}{2z - xy\cos(xyz)}$$

(b) tangent plane at $P = (1,1,0)$

$$\left.\frac{\partial z}{\partial x}\right|_P = \frac{0-2}{0-1} = 2 \qquad \left.\frac{\partial z}{\partial y}\right|_P = \frac{-2}{-1} = 2$$

$$\left.\frac{\partial z}{\partial x}\right|_P (x-1) + \left.\frac{\partial z}{\partial y}\right|_P (y-1) + (z-0) = 0$$

$$\boxed{2x + 2y + z = 4}$$

(c) $\left.\left\langle \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y} \right\rangle\right|_P = \langle 2, 2 \rangle$

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6. (a) $f(x,y) = \dfrac{2xy}{x^2+2y^2}$

If $y=x$ then $f(x,x) = \dfrac{2x\cdot x}{x^2+2x^2} = \dfrac{2x^2}{3x^2} = 2/3 \xrightarrow{x\to 0} 2/3$

If $x=0$ then

$f(0,y) = \dfrac{0}{2y^2} = 0 \xrightarrow[y\to 0]{} 0$

Hence the limit $\lim_{(x,y)\to(0,0)} f(x,y)$ D.N.E.

(b) $f(x,y) = \dfrac{x^4y^4\sin^2(x^2y^2)}{x^2+y^2}$

$$0 \le \frac{x^4y^4\sin^2(x^2y^2)}{x^2+y^2} \le \frac{x^4y^4}{x^2+y^2} \le \frac{x^4y^4}{x^2} = x^2y^4 \to 0$$

as $(x,y)\to(0,0)$

Hence $\lim_{(x,y)\to(0,0)} f(x,y) = 0$ by the Squeeze Theorem

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7. $f(x,y) = e^y(y^2-x^2)$

(a) $f_x(x,y) = -2x\,e^y$

$f_y(x,y) = e^y(y^2-x^2) + e^y \cdot 2y = e^y\left[y^2+2y-x^2\right]$

$\begin{cases} f_x=0 \\ f_y=0 \end{cases} \Leftrightarrow \begin{cases} -2x\,e^y=0 \Rightarrow x=0 \\ (y^2+2y-x^2)e^y=0 \Rightarrow y^2+2y=0 \end{cases}$

$y=0$ or $y=-2$

Two critical points: $(0,0)$ & $(0,-2)$

(b) $f_{xx} = -2e^y \qquad f_{xy} = -2x\,e^y = f_{yx}$

$f_{yy} = e^y\left[y^2+2y-x^2\right] + e^y\left[2y+2\right] = e^y\left[y^2+4y+2-x^2\right]$

At $\underline{(0,0)}$: $f_{xx}|_{(0,0)} = -2$, $f_{xy}|_{(0,0)} = 0$

$f_{yy}|_{(0,0)} = 2$ Hence $D|_{(0,0)} = (-2)\cdot 2 = -4 < 0$

so $(0,0)$ is a saddle point

At $\underline{(0,-2)}$ $f_{xx}|_{(0,-2)} = -2e^{-2} < 0$

$f_{xy}|_{(0,-2)} = 0 \qquad f_{yy}|_{(0,-2)} = e^{-2}\left[4-8+2\right] = -2e^{-2}$

$D|_{(0,-2)} = 4e^{-4} > 0$ so f has a maximum at $(0,-2)$

7 c) Absolute max & min on Disk $x^2+y^2\le 4$

$(0,0)$ & $(0,-2)\in$ Disk

$$f(0,0)=0$$
$$f(0,-2)=4e^{-2}$$

On the boundary: $x^2=4-y^2$

$$f(x,y)\big|_{x^2=4-y^2}=g(y)=e^y(2y^2-4)$$

$$\frac{dg}{dy}=e^y(2y^2-4)+e^y(\cdot 4y)=e^y\left[2y^2+4y-4\right]$$

$$\frac{dg}{dy}=0 \iff y^2+2y-2=0$$

$$\Delta=4+8=4\cdot 3 \qquad y=\frac{-2\pm 2\sqrt{3}}{2}$$

$y=-1+\sqrt{3}$ or $\underbrace{y=-1-\sqrt{3}}_{\text{no}}\notin$ Disk

$$g(-1+\sqrt{3})=e^{-1+\sqrt{3}}\left(2(-1+\sqrt{3})^2-4\right)=$$

$=e^{-1+\sqrt{3}}$(negative number) <0 absolute minimum

$$g(2)=f(0,2)=\cancel{e^y}\,e^2(2\cdot 4-4)=4e^2$$
$$g(-2)=f(0,-2)=e^{-2}(2\cdot 4-4)=4e^{-2}$$

absolute maximum

8. $f(x,y) = e^{xy}$ subject to $x^3+y^3=16$

$\nabla f = \langle e^{xy}\cdot y, e^{xy}\cdot x \rangle$

$\lambda \nabla g = \lambda \langle 3x^2, 3y^2 \rangle$ $\left\{ \begin{array}{l} \nabla f = \lambda \nabla g \\ \& \; x^3+y^3=16 \end{array} \right.$

$$\left\{ \begin{array}{l} y e^{xy} = \lambda \cdot 3x^2 \\ x e^{xy} = \lambda \cdot 3y^2 \\ x^3+y^3=16 \end{array} \right.$$

if $x=0 \Rightarrow y=0$
if $y=0 \Rightarrow x=0$
but then contradiction

Hence $x \neq 0$ & $y \neq 0$

$$\lambda = \frac{y e^{xy}}{3x^2} = \frac{x e^{xy}}{3y^2} \Rightarrow y^3 = x^3 \Rightarrow x = y$$

$x^3+y^3=16$
$2x^3=16 \Rightarrow x^3=8$ & $x=2=y$

Maximum value is at $(2,2)$ &
$f(2,2)=e^4$

Minimum value does not exist.