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University of Ottawa  
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**uOttawa**  
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School of Information  
Technology and Engineering

CSI 2101 Discrete Structures – Final Exam  
Instructor: Lucia Moura

April 18, 2010

19:00-22:00

Duration: 3hs

Closed book, no calculators

Last name: \_\_\_\_\_

First name: \_\_\_\_\_

Student number: \_\_\_\_\_

There are 8 questions and 100 marks total.

This exam paper should have 16 pages,  
including this cover page.

Theorems regarding recurrence relations are provided  
in pages 15-16.

1 – Predicate Logic - short answers	/ 10
2 – Induction 1	/ 10
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8 – Program correctness	/ 10
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Total	/ 100

# 1 Predicate Logic - short answers — 10 points

## Part A — 6 points

Suppose  $P(x, y)$  is a predicate and the universe for the variables  $x$  and  $y$  is  $\{1, 2, 3\}$ . Suppose  $P(1, 3)$ ,  $P(2, 1)$ ,  $P(2, 2)$ ,  $P(2, 3)$ ,  $P(3, 1)$ ,  $P(3, 2)$  are true, and  $P(x, y)$  is false otherwise.

Determine whether the following statements are true or false:

- [true/false]  $\forall x \exists y P(x, y)$
- [true/false]  $\exists x \forall y P(x, y)$
- [true/false]  $\neg \exists x \exists y (P(x, y) \wedge \neg P(y, x))$
- [true/false]  $\forall y \exists x (P(x, y) \rightarrow P(y, x))$
- [true/false]  $\forall x \forall y ((x \neq y) \rightarrow (P(x, y) \vee P(y, x)))$
- [true/false]  $\forall y \exists x ((x \leq y) \wedge P(x, y))$

**Part B — 4 points** Suppose the variable  $x$  represents people, and  
 $F(x)$ :  $x$  is friendly     $T(x)$ :  $x$  is tall     $A(x)$ :  $x$  is angry

Write the statement using these predicates and any needed quantifiers:

- Some tall angry people are friendly.
- If a person is friendly, then that person is not angry.

## 2 Induction 1 — 10 points

Use the principle of mathematical induction to prove that  $2|(n^2 + n)$  for all  $n \geq 0$ .  
(recall that the symbol “|” means “divides”)

### 3 Induction 2 — 10 points

We are given a chocolate bar with  $m \times n$  squares of chocolate, and our task is to divide it into  $mn$  individual squares. We are only allowed to split one piece of chocolate at a time using a vertical or a horizontal break.

For example, suppose that the chocolate bar is  $2 \times 2$ . The first split makes two pieces, both  $2 \times 1$ . Each of these pieces requires one more split to form single squares. This gives a total of three splits.

Use strong induction to conclude the following:

“To divide up a chocolate bar with  $m \times n$  squares, we need at most  $mn - 1$  splits.”

Hint: Use strong induction on  $k$ , the number of squares in the chocolate bar ( $k = mn$ ).

## 4 Number theory 1 — 10 points

**Part A — 5 points** Find the inverse of 21 modulo 44 using the extended Euclidean Algorithm.

**Part B — 5 points** Using the solution above, find all integer solutions to the following linear congruence:

$$21x \equiv 3 \pmod{44}.$$

## 5 Proof methods/number theory — 14 points

**Part A — 4 points** Let  $m$  and  $n$  be integers. Use a proof by **contraposition** to show that if  $mn$  is even then  $m$  is even or  $n$  is even.

**Part B — 4 points** Use a proof by **contradiction** to prove that at least one of the numbers  $a_1, a_2, \dots, a_n$  is greater than or equal to the average of these numbers,  $(a_1 + a_2 + \dots + a_n)/n$ .

**Part C — 6 points** Prove that if  $n$  is an odd positive integer, then  $n^2 \equiv 1 \pmod{8}$ .

(more space to solve question 5)

## 6 Recurrence relations — 20 points

For this question, you can refer to the theorems provided in pages 15-16.

### Part A — 10 points

Consider the recurrence relation  $a_n = 3a_{n-1} + 5^n$ .

- A. Write the associated homogeneous recurrence relation.
- B. Find the general solution to the associated homogeneous recurrence relation.
- C. Find a particular solution to the given recurrence relation.
- D. Write the general solution to the given recurrence relation.
- E. Find the particular solution to the given recurrence relation when  $a_0 = 1$ .

(space to continue solution...)

**Part B — 10 points**

Consider the following recursive algorithm:

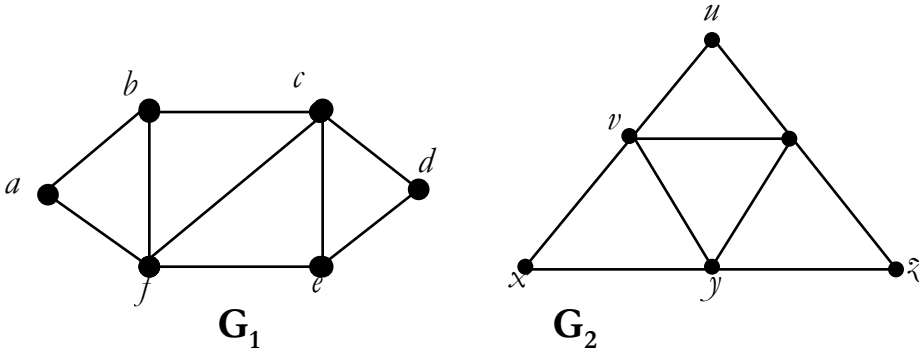
```
procedure LongIntegerMultiply( $X, Y, n$ :  $X$  and  $Y$  are  $n$ -bit integers,  $n$  is a power of 2)
  if  $n = 1$  then return  $X * Y$  /* line 1: single-bit multiplication */
  else
    split  $X$  into  $X_1, X_2$  and  $Y$  into  $Y_1, Y_2$  such that  $X = 2^{n/2}X_1 + X_2$  and  $Y = 2^{n/2}Y_1 + Y_2$ 
     $U \leftarrow$  LongIntegerMultiply( $X_1, Y_1, n/2$ )
     $V \leftarrow$  LongIntegerMultiply( $X_2, Y_2, n/2$ )
     $W \leftarrow$  LongIntegerMultiply( $X_1 - X_2, Y_1 - Y_2, n/2$ )
     $Z \leftarrow U + V - W$ 
    return  $2^n U + 2^{n/2} Z + V$ 
```

- A.** Set up a divide-and-conquer recurrence relation for the number of single-bit multiplications (done in line 1) required to compute the product of two  $n$ -bit integers  $X$  and  $Y$ , where  $n$  is a power of 2 (i.e.  $n = 2^k$  for some integer  $k$ ), using the algorithm above.

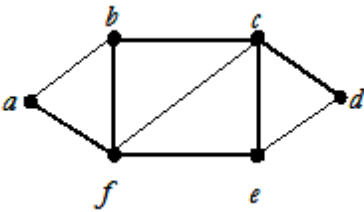
- B.** Use the recurrence relation above and the Master theorem to derive a big-O estimate for the number of single-bit multiplications used in the algorithm above.

## 7 Graphs — 16 points

**Part A — 4 points** Are the following graphs isomorphic? Explain your answer.



**Part B — 4 points** Consider the following graph

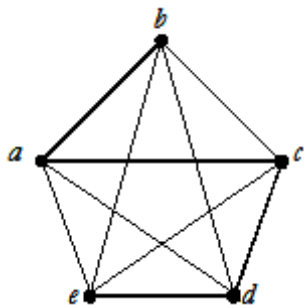


A. Does it have an Euler circuit? If yes, state it. If no, explain why.

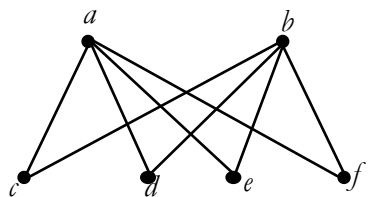
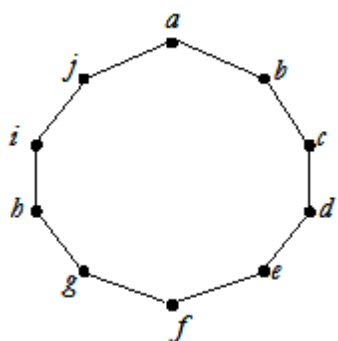
B. Does it have a Hamilton circuit? If yes, state it. If no, explain why.

## Part C — 6 points Graph colouring

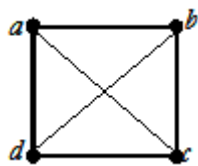
- Is the following graph 4-colourable? [yes/no] Justify:



- What is the chromatic number of each of the following graphs? Justify.



- Part D — 2 points Is the following graph planar? [yes/no] Justify:



## 8 Program correctness — 10 points

Consider the following program segment  $S$ :

```
 $i \leftarrow 1$   
 $total \leftarrow 1$   
while  $i < n$  do  
     $i \leftarrow i + 1$   
     $total \leftarrow total + i$   
endwhile
```

**Part A — 5 points** Let  $p$  be the proposition “ $total = \frac{i(i+1)}{2}$  and  $i \leq n$ ”. Prove that  $p$  is a loop invariant for the while loop.

**Part B — 5 points** Use program verification techniques to show that  $S$  is correct with respect to the initial assertion (precondition) “ $n \geq 1$ ” and the final assertion (postcondition) “ $total = \frac{n(n+1)}{2}$ ”. You may use the loop invariant in part A, even if you didn’t prove it.

(more space for question 8)

**Recurrence relation theorems:**

**MASTER THEOREM** Let  $f$  be an increasing function that satisfies the recurrence relation

$$f(n) = af(n/b) + cn^d$$

whenever  $n = b^k$ , where  $k$  is a positive integer,  $a \geq 1$ ,  $b$  is an integer greater than 1, and  $c$  and  $d$  are real numbers with  $c$  positive and  $d$  nonnegative. Then

$$f(n) \text{ is } \begin{cases} O(n^d) & \text{if } a < b^d, \\ O(n^d \log n) & \text{if } a = b^d, \\ O(n^{\log_b a}) & \text{if } a > b^d. \end{cases}$$

**THEOREM 1** Let  $c_1$  and  $c_2$  be real numbers. Suppose that  $r^2 - c_1r - c_2 = 0$  has two distinct roots  $r_1$  and  $r_2$ . Then the sequence  $\{a_n\}$  is a solution of the recurrence relation  $a_n = c_1a_{n-1} + c_2a_{n-2}$  if and only if  $a_n = \alpha_1r_1^n + \alpha_2r_2^n$  for  $n = 0, 1, 2, \dots$ , where  $\alpha_1$  and  $\alpha_2$  are constants.

**THEOREM 2** Let  $c_1$  and  $c_2$  be real numbers with  $c_2 \neq 0$ . Suppose that  $r^2 - c_1r - c_2 = 0$  has only one root  $r_0$ . A sequence  $\{a_n\}$  is a solution of the recurrence relation  $a_n = c_1a_{n-1} + c_2a_{n-2}$  if and only if  $a_n = \alpha_1r_0^n + \alpha_2nr_0^n$ , for  $n = 0, 1, 2, \dots$ , where  $\alpha_1$  and  $\alpha_2$  are constants.

**THEOREM 3** Let  $c_1, c_2, \dots, c_k$  be real numbers. Suppose that the characteristic equation

$$r^k - c_1r^{k-1} - \dots - c_k = 0$$

has  $k$  distinct roots  $r_1, r_2, \dots, r_k$ . Then a sequence  $\{a_n\}$  is a solution of the recurrence relation

$$a_n = c_1a_{n-1} + c_2a_{n-2} + \dots + c_ka_{n-k}$$

if and only if

$$a_n = \alpha_1r_1^n + \alpha_2r_2^n + \dots + \alpha_kr_k^n$$

for  $n = 0, 1, 2, \dots$ , where  $\alpha_1, \alpha_2, \dots, \alpha_k$  are constants.

## Recurrence relation theorems (cont'd):

**THEOREM 5** If  $\{a_n^{(p)}\}$  is a particular solution of the nonhomogeneous linear recurrence relation with constant coefficients

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n),$$

then every solution is of the form  $\{a_n^{(p)} + a_n^{(h)}\}$ , where  $\{a_n^{(h)}\}$  is a solution of the associated homogeneous recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}.$$

**THEOREM 6** Suppose that  $\{a_n\}$  satisfies the linear nonhomogeneous recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n),$$

where  $c_1, c_2, \dots, c_k$  are real numbers, and

$$F(n) = (b_t n^t + b_{t-1} n^{t-1} + \cdots + b_1 n + b_0) s^n,$$

where  $b_0, b_1, \dots, b_t$  and  $s$  are real numbers. When  $s$  is not a root of the characteristic equation of the associated linear homogeneous recurrence relation, there is a particular solution of the form

$$(p_t n^t + p_{t-1} n^{t-1} + \cdots + p_1 n + p_0) s^n.$$

When  $s$  is a root of this characteristic equation and its multiplicity is  $m$ , there is a particular solution of the form

$$n^m (p_t n^t + p_{t-1} n^{t-1} + \cdots + p_1 n + p_0) s^n.$$