MAT 1330

- Welcome! Introduction
- Go over syllabus
- Why math matters → 33.5 million Canadians in 2011. How many in 2021?
  → Drink 1 drink an hour for 5 hours. How long until sober again?
  → Given a patient (size, gender, weight...) how high a dose of medication is required?
  → Shape of a blood drop on the floor → where did it come from

⇒ Check out web resources: plus.maths.org ...
or try "why math matters" on google

- This class and next is review of the basics. You (should) have learned all this in school. And you will need it here. In math everything builds on previous things.
- If you play an instrument, you need to know the scales. Same here.
- So, before we go to the material and the summary notes:
  What does 0 say to 8?
- Go through cover page of Summary notes
- We don't really do percentage calculations here. I assume you know that. Teacher: This class is so miserable at math that 90% will fail.
  Student: But we are not that many.
- "Don't skip steps". The following review gives you an indication of which details I won't give later in the course.

Algebraic Manipulations

Simplify:

(a) $\dfrac{\sqrt{x^{1/2}\,y^5}}{x^5\;y^{1/4}}$

(b) $\dfrac{\left(x^{1/2}\,y^{1/3}\right)^{-1/2}}{x^2y^3}$

(c) $\dfrac{4+\frac{1}{x}}{\frac{5}{x}+2}$

(d) $\dfrac{z^{-1}-3^{-2}}{z^{-2}+3^{-1}}$

Rationalize the denominator:

(a) $\dfrac{1}{\sqrt{10}-3}$

(b) $\dfrac{\sqrt{6}-\sqrt{8}}{\sqrt{6}+\sqrt{8}}$

Solve for x:

(a) $2^{x+3}=16^{2x-1}$

(b) $2^{2x+3}=3^{4x-1}$

(c) $\dfrac{1}{x}+\dfrac{2}{y}=\dfrac{5}{z}$

(d) $\dfrac{2}{x}-\dfrac{6}{y}=\dfrac{z}{3}$

Solve for x:

(a) $\dfrac{1}{x}+\dfrac{1}{x^2}=1$

(b) $x=\sqrt{x+6}$

(c) $\dfrac{4x}{1+x}=3x-1$

(d) $\dfrac{4x}{x+1}=3$

Factor (a) x^3+1000

(b) $x^3+x^2+\frac{5}{4}x+3$

Hint: $(x+\frac{3}{2})$ is a factor

Find k such that $x^2+2kx+9k-8=0$ has only one solution.

Inequalities

(a) $\frac{x}{2} - 3 > 5$

(c) $\frac{x}{3} + 4 > -1$

(b) $\frac{2}{x} - 3 > 5$

(d) $\frac{3}{x} + 4 < -1$

Absolute value:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Solve for x:

(a) $|x^2 - 5| = 1$

(b) $|x^2 - 16| = 9$

(c) $\left|\frac{x}{2} - 3\right| > 5$

(d) $|12 - x^2| > 3$

- Now go and practice, practice, practice.

Absolute value and equalities

Definition : $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

Since the definition is in two pieces, equations with absolute values often require two cases. Let's go through a few simple examples.

Ex 1 : $|x-3| = 6$

There are two cases: $x-3 \geq 0$ and $x-3 < 0$.

a) Case: $x-3 \geq 0$, i.e. $x \geq 3$.

Then $|x-3| = x-3$ and so we need to solve

$x-3 = 6$ or $x = 9$.

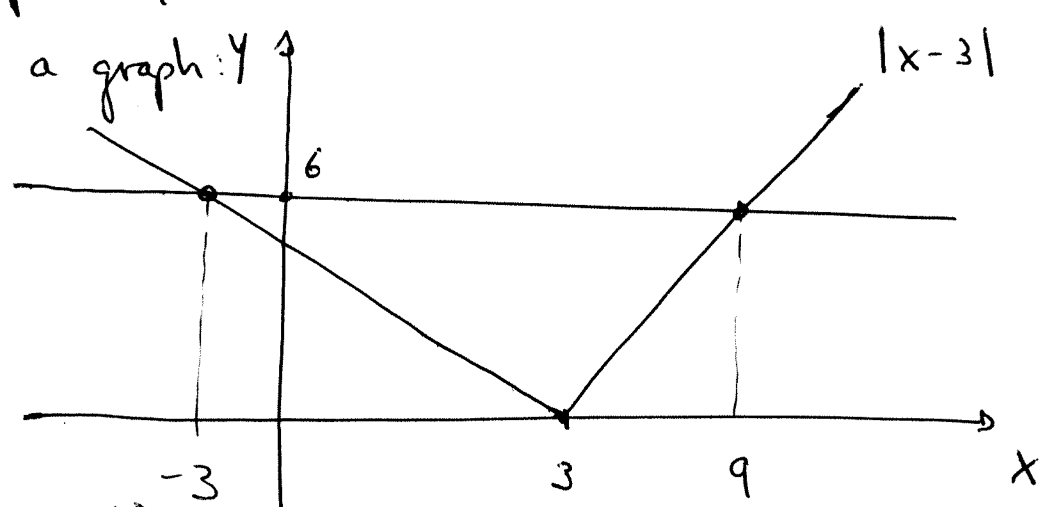
Now we need to check: $x \geq 3$ and $x = 9$ are indeed compatible, so $x = 9$ is a solution.

b) Case: $x-3 < 0$, i.e. $x < 3$

Then $|x-3| = -(x-3) = 3-x$, and so we need to solve $3-x = 6$ or $x = -3$

Again, we need to check: $x < 3$ and $x = -3$ are compatible, so $x = -3$ is a solution.

c) Let's look at a graph:



(not to scale!)

Ex 2: $|x^2-5|=1$

There are again two cases: $x^2-5 \geq 0$ and $x^2-5<0$

a) Case: $x^2-5 \geq 0$ i.e. $x^2 \geq 5$ or $(x \geq \sqrt{5}$ or $x \leq -\sqrt{5})$

Then $|x^2-5| = (x^2-5)$ and so we have to solve

$$x^2-5=1 \quad \text{or} \quad x^2=6 \quad \text{or} \quad x=\pm\sqrt{6}$$

Now we need to check: since $\sqrt{6}>\sqrt{5}$ and $-\sqrt{6}<-\sqrt{5}$ the two are compatible, and $x=\pm\sqrt{6}$ are two solutions

Let's put it into the original equality:

If $x=\pm\sqrt{6}$ then $x^2=6$ and $|x^2-5|=|6-5|=|1|=1$ ✓

b) Case: $x^2-5<0$ i.e. $x^2<5$ or $(-\sqrt{5}<x<\sqrt{5})$

Then $|x^2-5| = -(x^2-5) = 5-x^2$ and so we have to solve

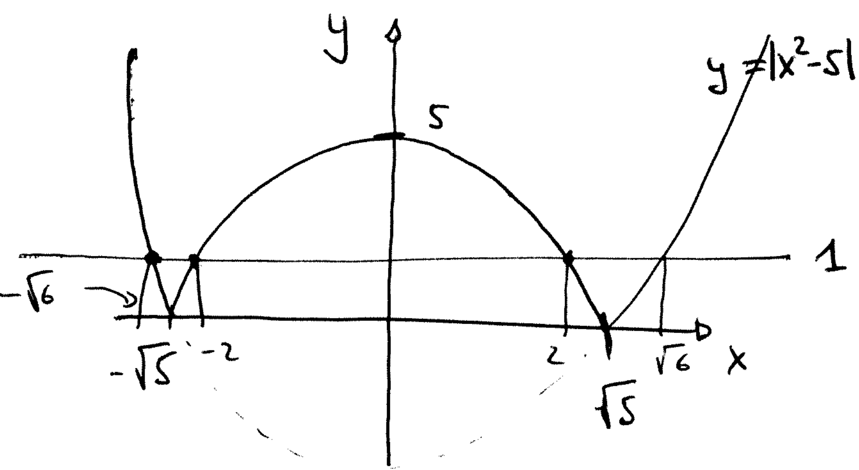
$$5-x^2=1 \quad \text{or} \quad x^2=4 \quad \text{or} \quad x=\pm 2$$

Now we need to check: $2<\sqrt{5}$ and $-2>-\sqrt{5}$,

So the two conditions are compatible, and $x=\pm 2$ are two solutions. Let's check:

$x=\pm 2 \Rightarrow x^2=4 \Rightarrow |x^2-5|=|4-5|=|-1|=1$ ✓

c) Let's look at a graph



The same ideas work with inequalities.

Ex 3: $|x^2-12| > 3$

There are two cases: $x^2-12 \geq 0$ and $x^2-12 < 0$

a) Case: $x^2-12 \geq 0$ or $x^2 \geq 12$ or $(x > \sqrt{12}$ or $x < -\sqrt{12})$

Then $|x^2-12| = x^2-12$ and we have to solve

$x^2-12 > 3$ or $x^2 > 15$ or $(x > \sqrt{15}$ or $x < -\sqrt{15})$

Now we check: $x > \sqrt{15} > \sqrt{12}$ and $-\sqrt{15} < -\sqrt{12}$, so the two conditions are compatible, and

$x > \sqrt{15}$ or $x < -\sqrt{15}$ is part of the solution.

b) Case: $x^2-12 < 0$ or $x^2 < 12$ or $(-\sqrt{12} < x < \sqrt{12})$.

Then $|x^2-12| = 12-x^2$ and we have to solve

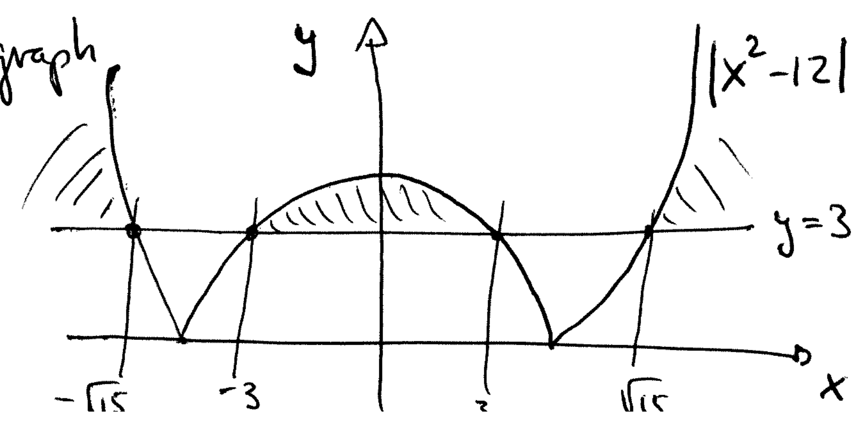
$12-x^2 > 3$ or $x^2 < 9$ (note the sign!)

Now $x^2 < 9$ gives $-3 < x < 3$.

We need to check! Since $\sqrt{12} > 3$, the two conditions are compatible and $-3 < x < 3$ is another part of the solution.

c) The complete solution is $\{x < -\sqrt{15}\} \cup \{-3 < x < 3\} \cup \{x > \sqrt{15}\}$

Let's look at the graph



Not everything has to have absolute values to require cases.

Ex: $\frac{3}{x} + 4 < -1$

First subtract 4 : $\frac{3}{x} < -5$

Then divide by -5 (!) : $\frac{3}{-5x} > 1$

Now look at cases:

Case $x > 0$: Multiply $-\frac{3}{5} > x$ or $x < -\frac{3}{5}$

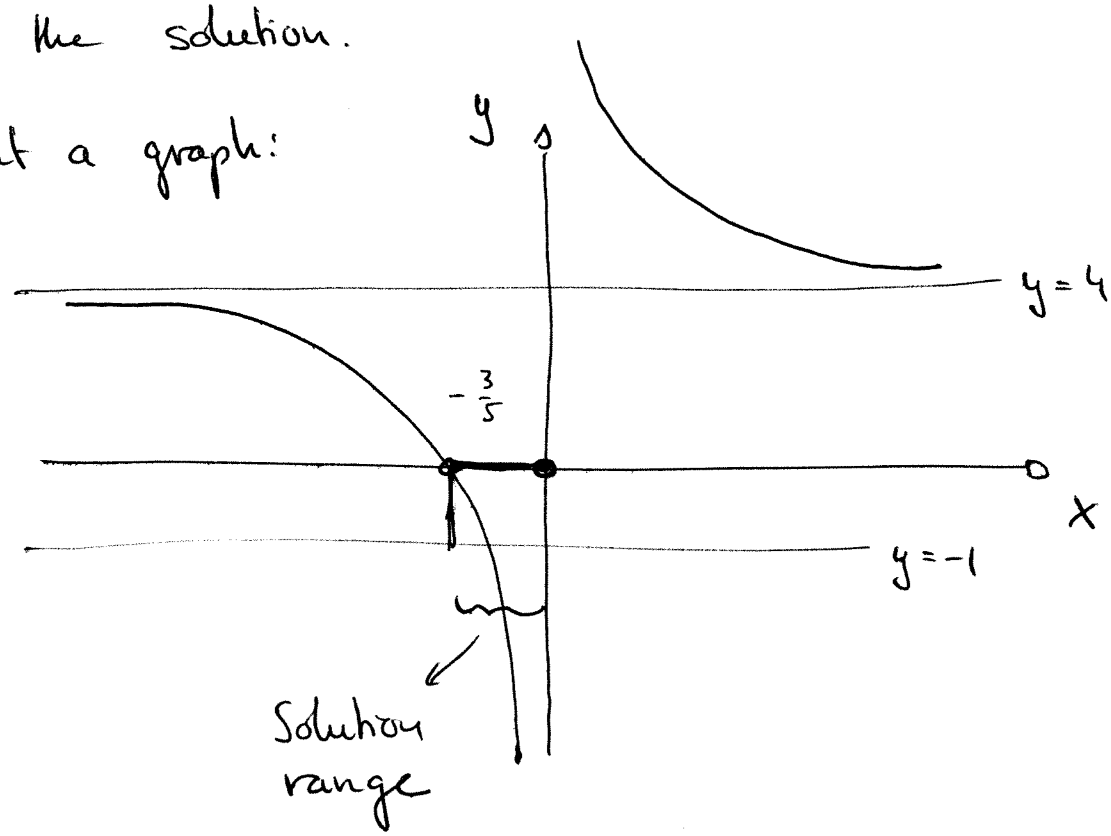
The two conditions $x > 0$ and $x < -\frac{3}{5}$ are not compatible. There is no solution.

Case $x < 0$: Multiply : $-\frac{3}{5} < x$ or $x > -\frac{3}{5}$

The two conditions $x < 0$ and $x > -\frac{3}{5}$ can be combined to give $-\frac{3}{5} < x < 0$.

This is the solution.

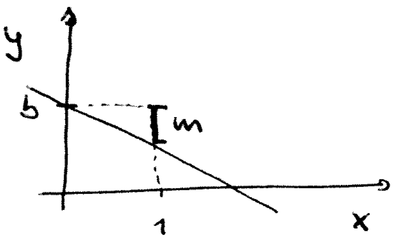
Let's look at a graph:



MAT 1330 Review of functions

- Notation: $y = f(x)$, domain, range, assignment, name

- Polynomial functions: <u>linear</u> $y = mx + b$



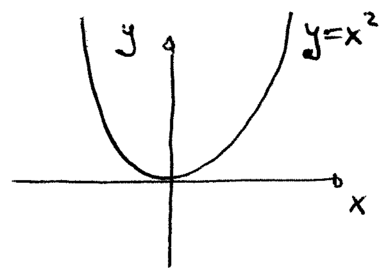
Find the equation of the line

(a) through $(0,1)$ and $(2,0)$

(b) through $(-1,3)$ and slope 2

(c) with x-intercept 5 and slope -1

- Polynomial functions: <u>quadratic</u> $y = ax^2 + bx + c$



Find zeros to graph parabolas (see page ②)

Find all values of k such that

$y = x^2 + 2kx + 13k - 40$ has exactly one zero.

- Polynomial functions of higher degree: factor to find zeros.

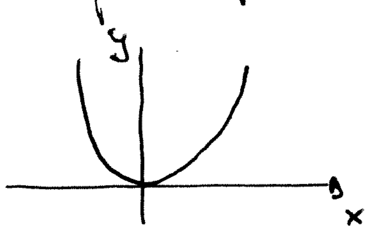
  long division (see page ②)

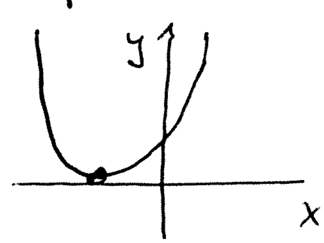
- Characteristics of functions:

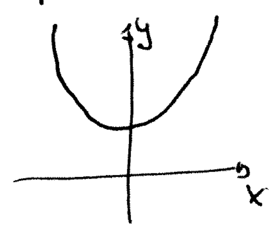
(a) symmetry: even if $f(x) = f(-x)$, odd if $f(x) = -f(-x)$

(b) transformations: $y = f(ax + b) + c$

Example: $f(x) = x^2$ $\quad f(x+1)$ $\quad f(x) + 1$







(c) composition: $(f \circ g)(x) = f(g(x))$

Example: $f(x) = x^2$, $g(x) = x+3$

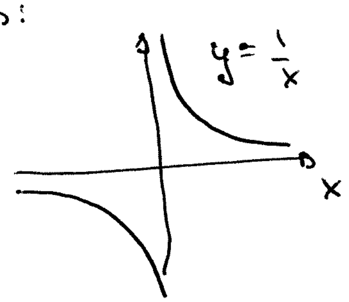
$$f(g(x)) = f(x+3) = (x+3)^2, \quad g(f(x)) = g(x^2) = x^2+3$$

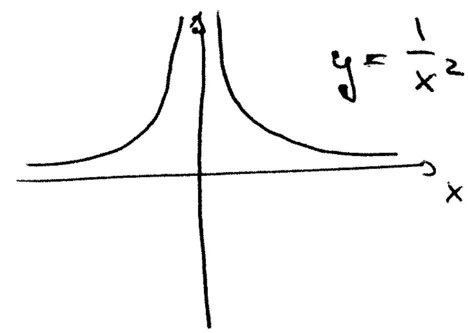
- Rational functions. Functions that are ratios of polynomials

  Important: Find the domain of definition

  Example: $f(x) = \dfrac{x+1}{x^2-2}$ $\qquad$ $g(x) = \dfrac{2}{x-1} + \dfrac{5}{x^2+3}$

  Examples:

- 

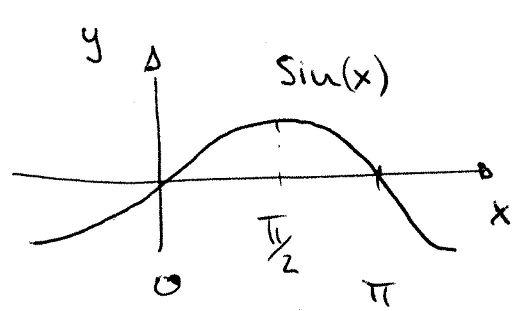


- Root functions. Example: $y = \sqrt{x-5}$
- Absolute value function. Example $y = |x|$,
- Trigonometric functions: $y = \sin(x)$, $y = \cos(x)$, $y = \tan(x) = \dfrac{\sin(x)}{\cos(x)}$

  $\rightarrow$ $\sin^2(x) + \cos^2(x) = 1$

  $\rightarrow$ amplitude, mean, period, phase

$$y = f(x) = M + A\cos\left(\frac{2\pi}{T}(x-p)\right)$$



- Exponential and logarithm functions

Base $a>0$: $f(x) = a^x$ special case $f(x) = e^x$

Laws of exponents: $a^x a^y = a^{x+y}$ $a^1 = a$ $e \approx 2.718\ldots$ Euler

$(a^x)^y = a^{xy}$ $a^0 = 1$

$a^{-x} = \frac{1}{a^x}$

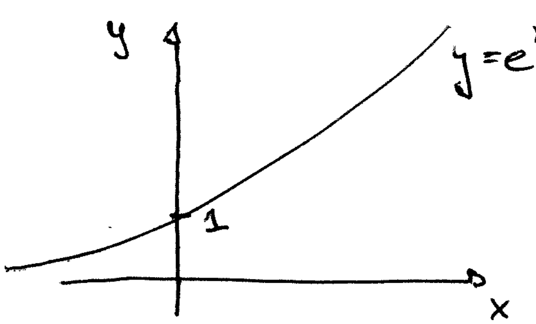
What is the domain of $f(x) = e^x$? What is the range?

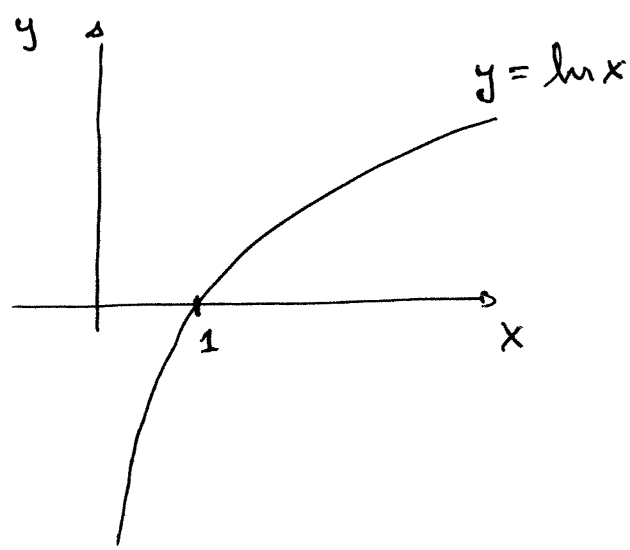
Logarithm base $a>0$: $f(x) = \log_a x = y$ if $a^y = x$

Laws: Special case: $\log_e x = \ln(x)$

$\ln(xy) = \ln(x) + \ln(y)$

$\ln(x^p) = p\ln(x)$





- Inverse functions:

f^{-1} is the inverse of f if: $f(f^{-1}(x)) = x = f^{-1}(f(x))$

Example: $f(x) = e^x \Rightarrow f^{-1}(x) = \ln(x)$

$f(x) = x^2,\ x>0 \Rightarrow f^{-1}(x) = \sqrt{x}$

Things in nature often change. Individuals grow and die; Population numbers change; individuals move in space; a state of an individual changes (e.g. disease); wounds heal (hopefully); hearts beat regularly (hopefully); the immune system works (hopefully)...

→ We want to be able to predict future states from present state, given certain mechanisms.

E.g. if we know how organisms growth rate depends on temperature, we can predict future developments under climate change.

→ Experiments can tell us sometimes what the future could bring, but
- costly; • risky; • large time requirements.

→ Mathematical models are invaluable tools to help prediction
E.g. weather, stock market, species harvesting and management.

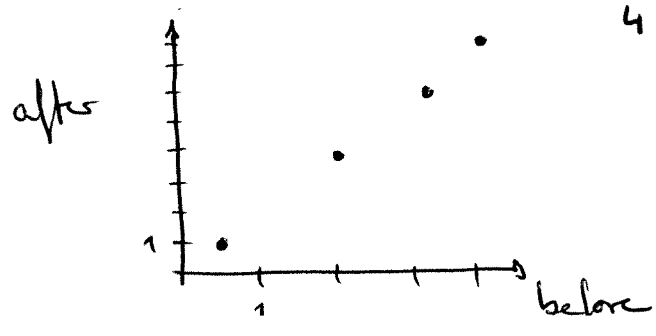
→ 3-step process:
1) from life science to math (modeling)
2) mathematical understanding (analysis)
3) from math to life sciences (interpretation)

This course will teach you some of each of these three steps but mostly step 2).

Example: Growth of Bacteria

Experiment in petridishes:
24 hour growth period
measured in cm^2 (area)

dish	before	after
1	3	6
2	0.5	1
3	2	4
4	4	8



Best guess: The bacteria double in 24 hours.

So, if I write this as a formula: x_{today} area covered by bacteria today

Then $$x_{\text{tomorrow}} = 2\, x_{\text{today}}$$

My first, simple model. But only one day. How can we predict further into the future? Answer: repeat!

$$x_{\text{two days}} = 2 \cdot x_{\text{tomorrow}} = 2 \cdot 2 \cdot x_{\text{today}} = 4\, x_{\text{today}}$$

$$x_{\text{three days}} = \ldots\ldots = 8\, x_{\text{today}}$$

Let us formalize this.

Definition: A discrete-time dynamical system (DTDS) consists of:
- a quantity whose change is tracked at time t: x_t
- a time step
- an updating function, that describes the change during a single time step: f

The DTDS then is $$\boxed{x_{t+1} = f(x_t)}$$

Example (Bacteria): $f(x) = 2x$; x: area of bacteria
time step: 1 day

DTDS: $$x_{t+1} = f(x_t) = 2x_t$$

Example (Tree hight): Bamboo is one of the fastest growing plants on earth with a rate of 3 cm/hour! x: length of bamboo in cm, time step: hour, $f(x) = x+3$ updating function

DTDS: $$x_{t+1} = x_t + 3$$

Example (medication): Daily dose of medication adds to the amount in the body; body clears the drug between intake.

time step: 1 day; quantity: x_t amount of drug in body in mg

updating function: $f(x) = ?$

Assume that we measure immediately after intake. Then

$$x_t \xrightarrow{\text{clearing}} r x_t \xrightarrow{\text{intake}} r x_t + c \qquad c>0,\quad 0 \le r < 1$$

r: fraction remaining after 24 hours. $(1-r)$ clearing rate.

So $f(x) = rx + c$ is the updating function and $x_{t+1} = r x_t + c$ is the DTDS.

Exercise: Write the DTDS for medication if measurements are taken immediately before intake.

Exercise: Assume you borrow \$1,000, and agree to pay back \$50 per month. The bank charges 0.5% interest per month. Write the updating function and the DTDS for x_t, the amount that you owe immediately after making the t^{th} payment.

Other uses of the updating function

Math		Life sciences
composition:	$(f \circ f)(x) = f(f(x))$	two time steps: $x_{t+2} = (f \circ f)(x_t)$
inverse:	$x = f^{-1}(y)$	previous time: $x_t = f^{-1}(x_{t+1})$

Exercise: Consider $f(x) = \frac{1}{2}x + 2$

Calculate the two-time steps map: $f \circ f$

Calculate the previous time-step map: f^{-1}

Okay, now that we know what a DTDS is, let's think about what a "solution" of a DTDS should be. Remember, the goal was to predict the future from the present and the short term mechanism. Maybe the "solution" should then be the entire future, i.e. a whole set, or sequence.

<u>Definition</u>: The <u>solution</u> of the DTDS $x_{t+1} = f(x_t)$ with <u>initial value</u> x_0 is the sequence $\{x_0, x_1, x_2, x_3, \dots\}$ where $x_{t+1} = f(x_t)$.

<u>Example (bacteria)</u>: The solution of $x_{t+1} = 2x_t$ with $x_0 = \frac{1}{2}$ is

$$\{\tfrac{1}{2}, 1, 2, 4, 8, \dots\}.$$

<u>Example (bamboo)</u>: The solution of $x_{t+1} = x_t + 3$ with $x_0 = 0$ is

$$\{0, 3, 6, 9, 15, \dots\}$$

<u>Note</u>: A solution is not a single number, not two numbers, but an entire sequence, i.e. infinitly many numbers!

A general solution formula for a DTDS with a <u>linear</u> updating function.

Assume: $x_{t+1} = f(x_t) = r x_t + c$

Then we calculate:

$$x_0$$
$$x_1 = r x_0 + c$$
$$x_2 = r(r x_0 + c) + c = r^2 x_0 + c(r+1)$$
$$x_3 = r(r^2 x_0 + c(r+1)) = \dots = r^3 x_0 + c(r^2 + r + 1)$$
$$x_4 = \dots = r^4 x_0 + c(r^3 + r^2 + r + 1)$$

Now we guess the pattern:

$$\boxed{x_t = r^t x_0 + c\left(r^{t-1} + r^{t-2} + \dots + r^2 + r + 1\right)} \quad \circledast$$

<u>Note</u>: The formula ⊛ allows us to calculate x_t <u>without</u> first calculating $x_1, x_2, x_3, \dots x_{t-1}$. !

<u>Note</u>: Formula ⊛ still requires us to calculate a sum of t terms, namely $r^{t-1} + r^{t-2} + \dots + r^2 + r + 1$.

But we can simplify:

1) If $r = 1$, then $r^t = 1$ for all t and so ⊛ becomes

$$\boxed{x_t = x_0 + ct} \qquad (**)$$

2) If $r \neq 1$ then the geometric series says

$$1 + r + r^2 + \dots + r^{t-1} = \frac{r^t - 1}{r - 1},$$ and so ⊛ becomes

$$\boxed{x_t = r^t x_0 + \frac{r^t - 1}{r - 1} c} \qquad (***)$$

After a little more algebraic manipulations:

$$\frac{r^t - 1}{r - 1} = \frac{r^t}{r-1} - \frac{1}{r-1} = -\frac{r^t}{1-r} + \frac{1}{1-r}$$ we get

$$x_t = r^t\left(x_0 - \frac{c}{1-r}\right) + \frac{c}{1-r}$$

So, if we set

$$x^* = \frac{c}{1-r}$$ then (***) becomes

$$\boxed{x_t = r^t(x_0 - x^*) + x^*} \qquad r \neq 1$$

<u>Example</u>: $x_{t+1} = \frac{1}{2}x_t + 2 \Rightarrow r = \frac{1}{2} \neq 1,\ c = 2 \Rightarrow x^* = 4 \Rightarrow x_t = \left(\frac{1}{2}\right)^t (x_0 - 4) + 4$

DTDS II

Recall: DTDS is $x_{t+1} = f(x_t)$. Explicit solution if f is linear.

Goal for today: Visualize the behavior of solutions, even if f is not linear.

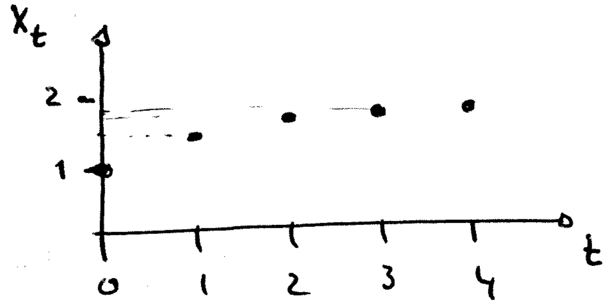
Method 1: First calculate the solution, then plot it.

Example: $x_{t+1} = \frac{1}{2} x_t + 1$; $x_0 = 1$

Recall $\begin{cases} x^* = 2 \\ x_t = \left(\frac{1}{2}\right)^t (x_0 - x^*) + x^* \end{cases}$

Calculate: $x_1 = 1.5$, $x_2 = 1.75$, $x_3 = 1.875$, $x_4 = \dots$

Plot:

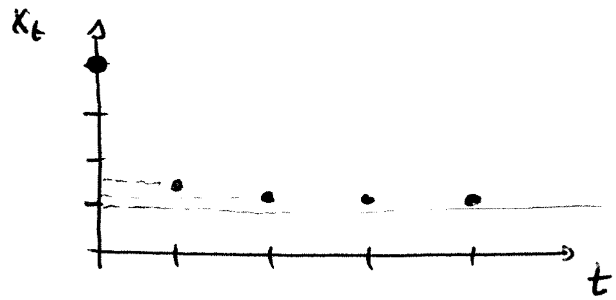


note: only for $t = 1, 2, 3, \dots$ not in between.

Example: $x_{t+1} = \dfrac{2x_t}{1 + x_t}$ $\qquad x_0 = 4$

Calculate: $x_1 = \frac{8}{5} = 1.6$, $x_2 \approx 1.23$, $x_3 = 1.10$, $x_4 = 1.05$

Plot



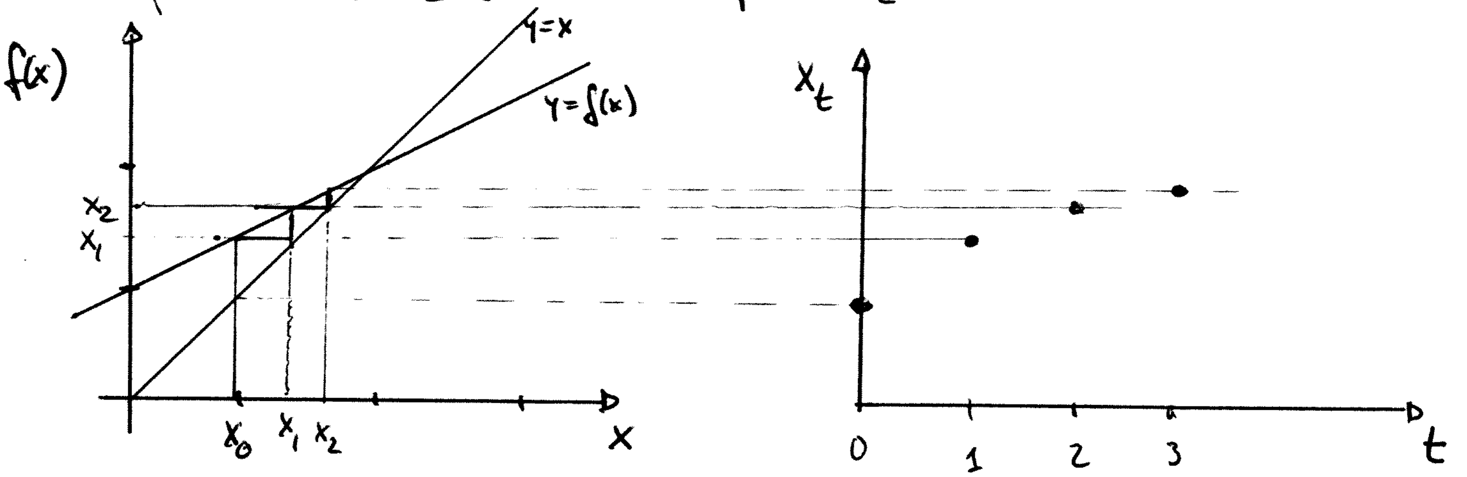
This method is possible and tedious.

Well suited for computers, but not so well for us.

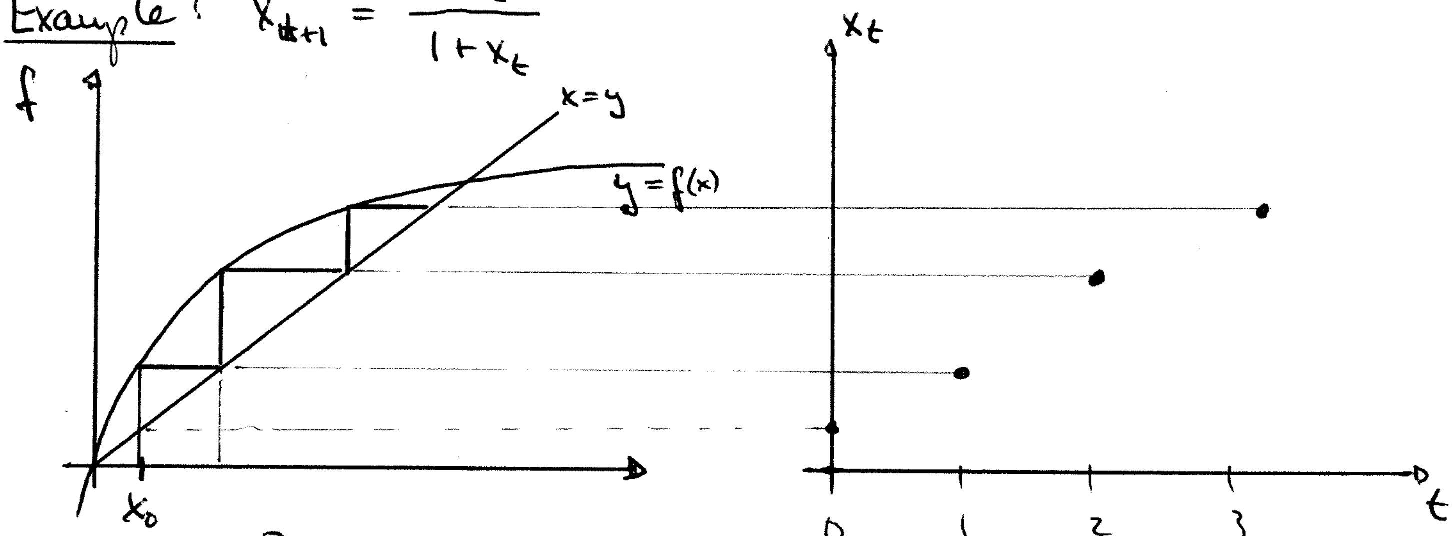
Method 2: Recipe

1) Graph updating function and diagonal
2) Start with x_0 and go vertically to x_1
3) Go horizontally to diagonal
4) Repeat 2) and 3)
5) Draw (t, x_t) if required

<u>Example</u>: $x_{t+1} = \frac{1}{2}x_t + 1$, $\quad f(x) = \frac{1}{2}x + 1$

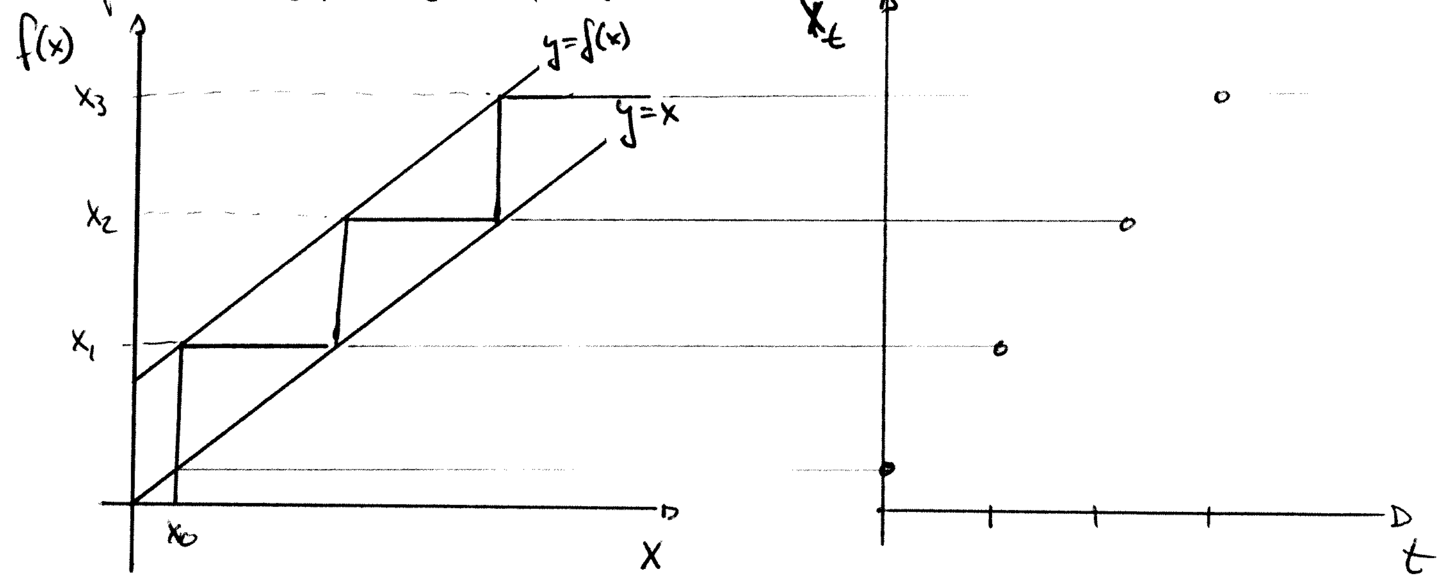


<u>Example</u>: $x_{t+1} = \dfrac{2x_t}{1+x_t}$



$$f(x) = \frac{2x}{1+x}$$

<u>Example</u>: $x_{t+1} = x_t + 1$, $\quad f(x) = x + 1$



<u>Observation</u>: If the graph of f intersects the diagonal then this point does not change when we cobweb from there. These points are very important, mathematically and for the application. They deserve a special name.

<u>Definition</u>: A point x^* is called <u>steady state</u> (or fixed point or equilibrium of the DTDS $x_{t+1} = f(x_t)$ if $f(x^*) = x^*$. Here, the dynamics do not change over time. [plural: equilibria]

<u>Example</u>: $x_{t+1} = \frac{1}{2}x_t + 1$. Find an equilibrium.

What to do? We need to replace x_{t+1} and x_t with x^* and solve for x^*.

$$x^* = \frac{1}{2}x^* + 1 \Rightarrow x^* - \frac{1}{2}x^* = 1 \Rightarrow \frac{1}{2}x^* = 1 \Rightarrow x^* = 2$$

<u>Example</u>: $x_{t+1} = \frac{2x_t}{1+x_t}$. Find all steady states.

Solve: $x^* = \frac{2x^*}{1+x^*}$.

$$\Rightarrow x^*(1+x^*) = 2x^*$$
$$\Rightarrow x^{*2} - x^* = 0$$
$$\Rightarrow x^* = 1 \text{ or } x^* = 0$$

<u>Example</u>: $x_{t+1} = x_t + 1$. Find all fixed points

Solve $x^* = x^* + 1 \Rightarrow$ no solution.

<u>Example</u>: $x_{t+1} = rx_t + c$. Find all fixed points

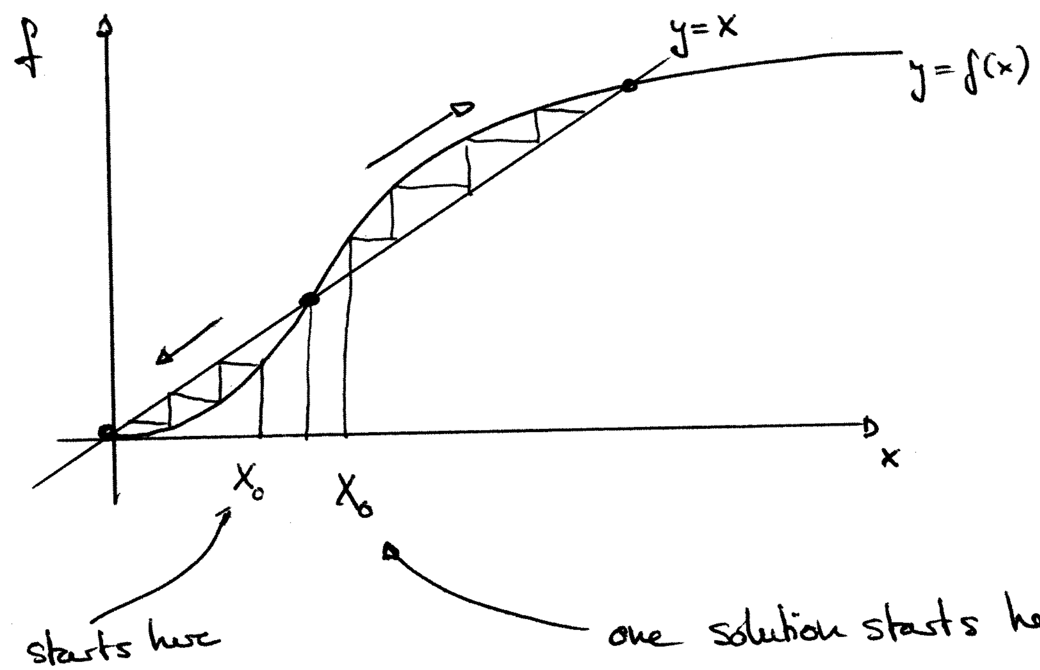
$$x^* = rx^* + c \Rightarrow x^*(1-r) = c \Rightarrow x^* = \frac{c}{1-r}$$

This is the same expression as on page (11)

For all examples: draw pictures!

Observation: Sometimes solutions approach an equilibrium, Sometimes solutions move away from an equilibrium.

Example: $x_{t+1} = \dfrac{4x_t^2}{1+x_t^2}$, $f(x) = \dfrac{4x^2}{1+x^2}$



The middle fixed point does not get approached, but the smallest and the largest do.

Definition: A fixed point is stable if all nearby solutions approach it, and unstable if at least one nearby solution does not approach it.

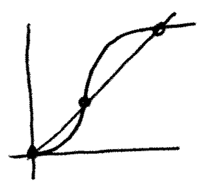
Exercise: Calculate the fixed points of $x_{t+1} = \dfrac{4x_t^2}{1+x_t^2}$ explicitly

Exercise: Which of the fixed points on p.13 are stable?

Exercise: Do a cobweb of $x_{t+1} = -\frac{1}{2}x_t + 2$

Explore: Use the excel file provided on the course website to explore cobwebbing and graphing. (LinearDTDS.xls)

Recall: DTDS $x_{t+1} = f(x_t)$. Fixed point: $x^* = f(x^*)$



Two kinds of fixed points: stable and unstable.

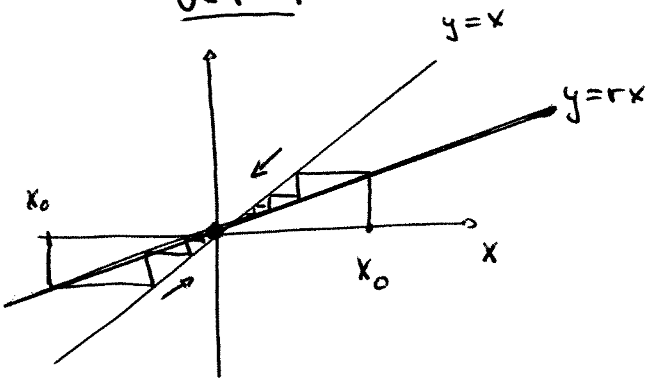
Stable: All nearby solutions approach x^*

Unstable: At least one nearby solution does not approach x^*

Goal: Study stability; use more examples / applications.

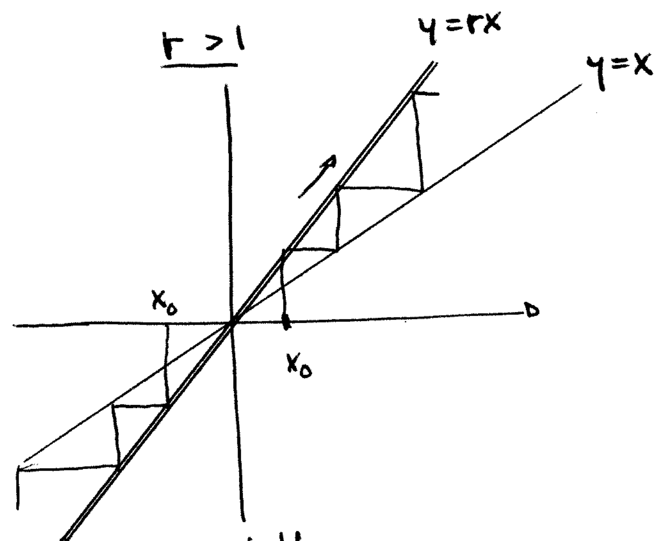
As usual: Start simple: $x_{t+1} = r x_t$. $x^* = 0$

$0 < r < 1$



stable

$r > 1$

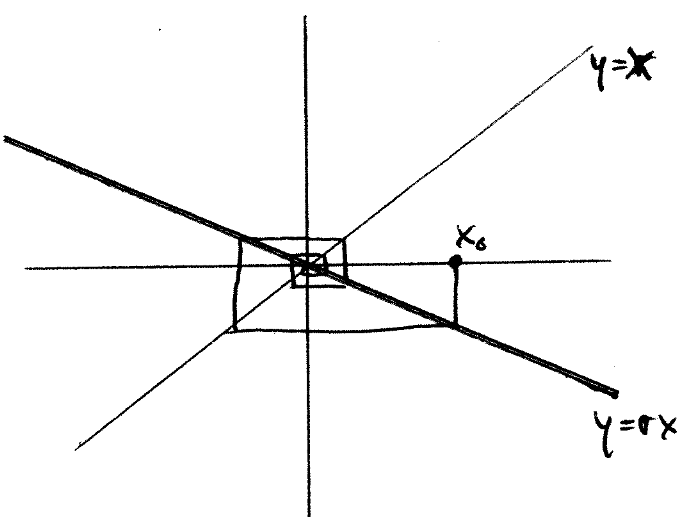


unstable

Note: To find stability via cobwebbing, you need to check at least two different initial conditions, one $x_0 > x^*$ and one $x_0 < x^*$

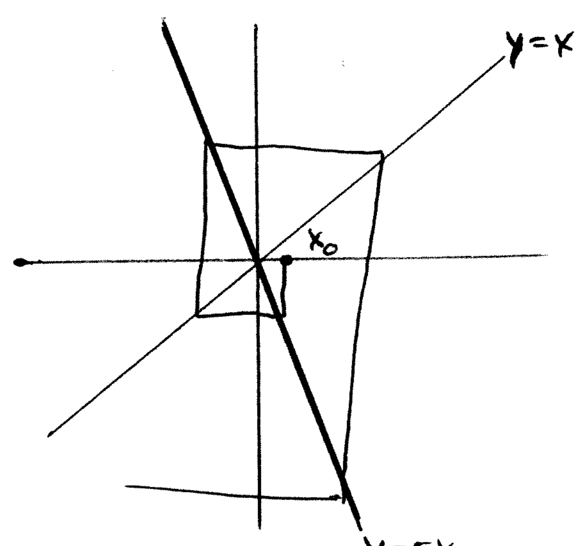
There is more

$-1 < r < 0$



Stable

$r < -1$



unstable

Can we guess stability? $-1<r<1$ stable

$|r|>1$ unstable

<u>Theorem:</u> Let $x_{t+1} = rx_t + c$. Then $x^* = \frac{c}{1-r}$ is stable if $|r|<1$ and unstable if $|r|>1$.

Why? Remember, we have the solution formula:

$$x_t = r^t(x_0 - x^*) + x^*.$$

If $|r|<1$ then r^t approaches zero as t approaches ∞, and so x_t approaches x^*.

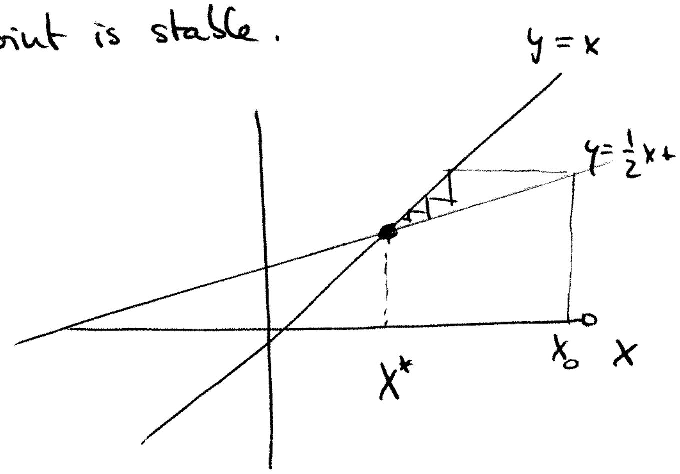
If $|r|>1$ then $|r^t|$ approaches infinity and not zero.

<u>Note:</u> We talk about "approaching" without formal definition. We will formalize this concept next class.

So, we know when the fixed point of a linear DTDS is stable. Why is this important? Because it tells us something about the long-term behavior of the system.

<u>Example:</u> Consider the medication model $x_{t+1} = \frac{1}{2}x_t + 1$ with constant dose 1. The fixed point is $x^* = 2$. Since $\frac{1}{2} = r < 1$, this fixed point is stable.

Therefore: If we continue this regular dose, the concentration will be close to $x^* = 2$.



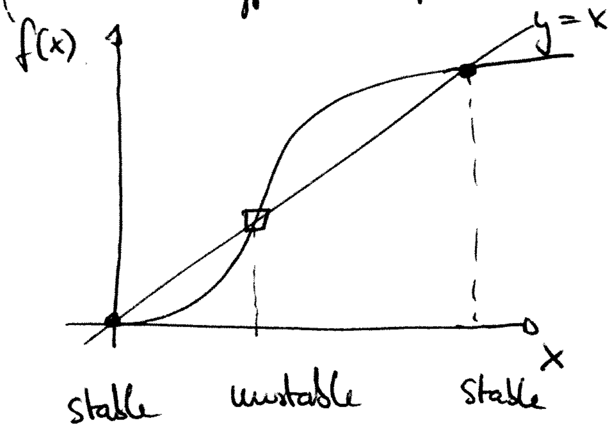
<u>Exercise:</u> Find the necessary dose to get a steady state of $x^* = 3$

Stability in nonlinear models

Why is it important? → Only stable states are visible in nature
→ Long-term behavior

Simple criterion: Not yet. Need more theory.

Example: Allee effect (see p15) $x_{t+1} = f(x_t) = \frac{4x_t^2}{1+x_t^2}$



Observe:

1) If the graph of f crosses the diagonal from below to above, then x^* is unstable

2) If the graph of f crosses from above to below with positive slope, then x^* is stable.

(see AlleeDTDS.xls)

Note: Eventually, we will need to study the slope of a nonlinear function to help identify stability. → Need more math tools first.

Example: Alcohol dynamics

Alcohol absorption in the body depends on alcohol level!
larger amount in the body ⇒ smaller fraction absorbed.

Pure absorpion: $c_{t+1} = c_t - \underbrace{r(c_t)}_{0<\cdot<1} c_t$ fraction absorbed.

Empirical: $r(c) = \frac{10}{4+c}$ when $c > 6$

Then $c_{t+1} = \left(1 - \frac{10}{4+c_t}\right) c_t$

c_t : alcohol concentration measured every hour.

Now assume the person drinks some per hour, then

$$c_{t+1} = c_t - r(c_t)c_t + d$$

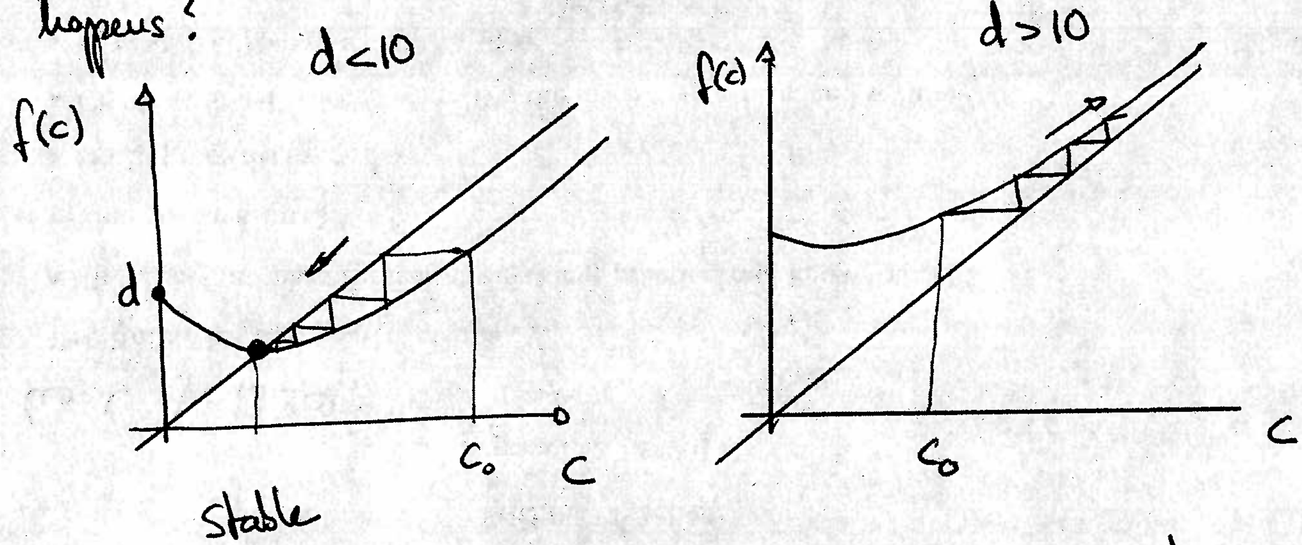
$$= \left(1 - \frac{10}{4+c_t}\right)c_t + d = f(c)$$

Steady state: $c^* = c^* - r(c^*)c^* + d$

or $\frac{10c^*}{4+c^*} = d \Rightarrow 10c^* = d(4+c^*) \Rightarrow (10-d)c^* = 4d$

$\Rightarrow c^* = \frac{4d}{10-d}$ Note $c^* > 0$ only when $d < 10$

What happens?



no steady state exists for $d > 10$. The body cannot absorb the alcohol as fast as it comes in.

Example

Caffeine: Absorption for caffeine is essentially independent of concentration: $c_{t+1} = 0.87c_t + d$

→ There is always a steady state: $c^* = \frac{d}{0.13}$

→ This state is stable.

⇒ The more one drinks, the more the concentration in the body, but at constant drinking of coffee, it levels off and stabilizes.

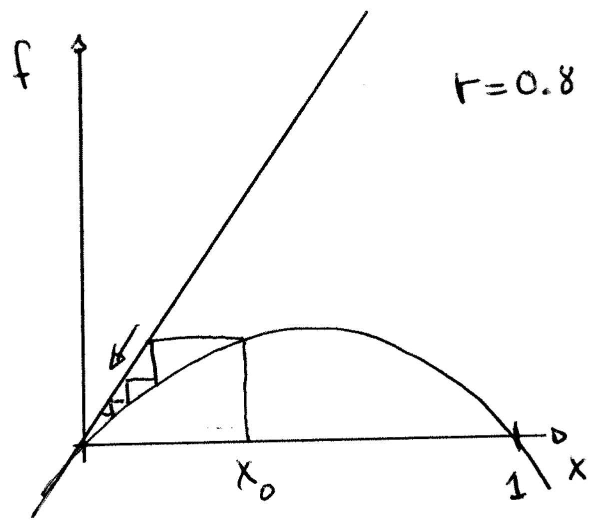
Example: The logistic equation (very famous!)

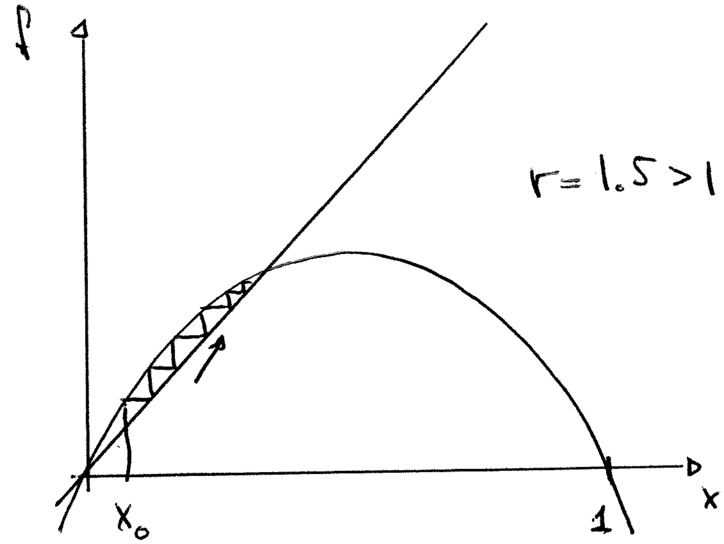
$$x_{t+1} = r x_t (1 - x_t) = f(x_t) \qquad 0 < r < 4 \,!$$

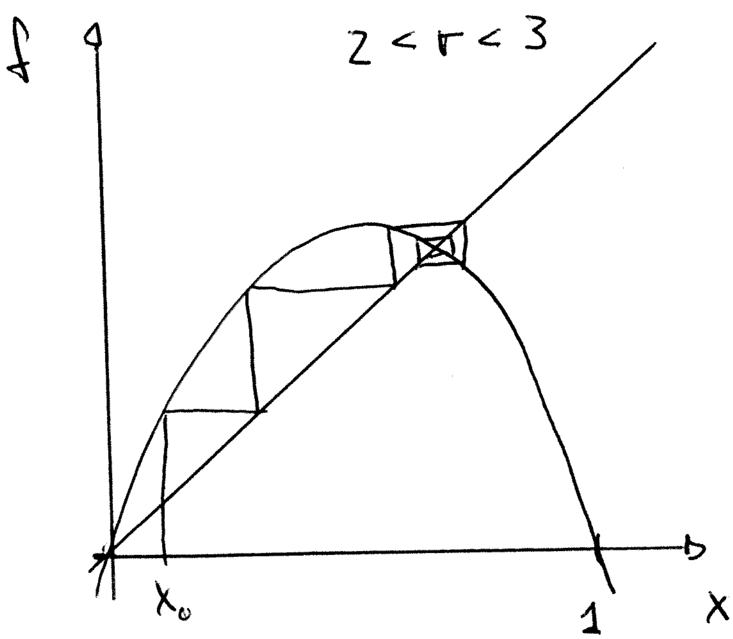
Interpretation: the per capita growth rate $\frac{x_{t+1}}{x_t} = r(1 - x_t)$ declines with density $\Rightarrow$ intraspecific competition

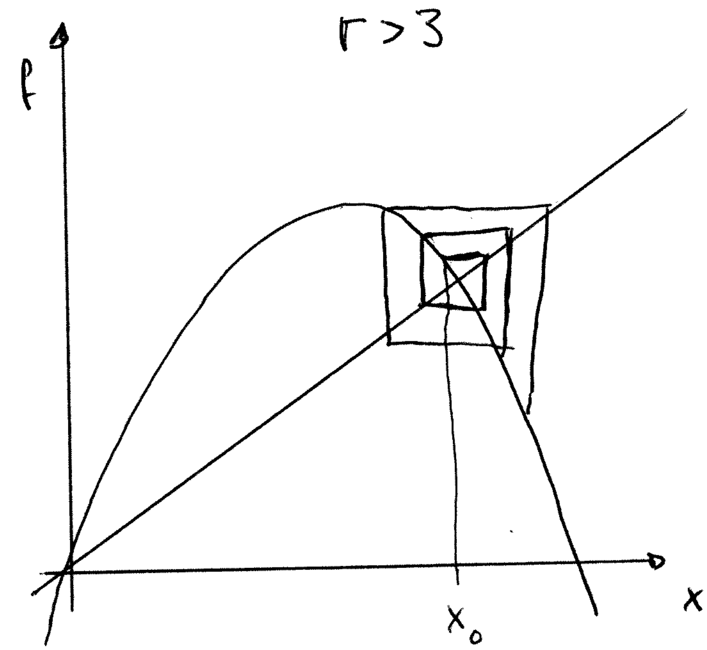
Fixed point: $x^* = r x^* (1 - x^*)$ $\rightarrow x_1^* = 0$

$\rightarrow x_2^* = 1 - \frac{1}{r} = \frac{r-1}{r}$









Try it out: LogisticDTDS.xls

Stability depends on r, and somehow on the "slope" of f at x^*

$\rightarrow$ need to define slope!

Additional Material: Modeling Heartbeat

Goal: Understand regular and irregular heart-beat.

Biology: SA-node (sinoatrial) is the pacemaker, sends regular signal.

AV-node (atrioventricular) is the controller.

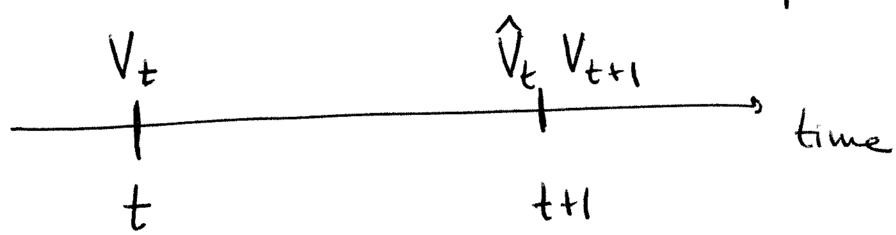
If potential V is low enough, it lets signal pass.

If potential V is too high, it blocks signal.

Heart beats if and when signal comes through.

State of the system: potential V_t in the heart after t-th signal from SA

Time t counts the regular signals from SA node.



From t to $t+1$ the potential decreases according to $\hat{V}_t = e^{-\alpha} V_t$

where α is the "decay factor" between beats

Then, $V_{t+1} = \hat{V}_t$ if the signal does not come through, i.e. if $\hat{V}_t$ is too large

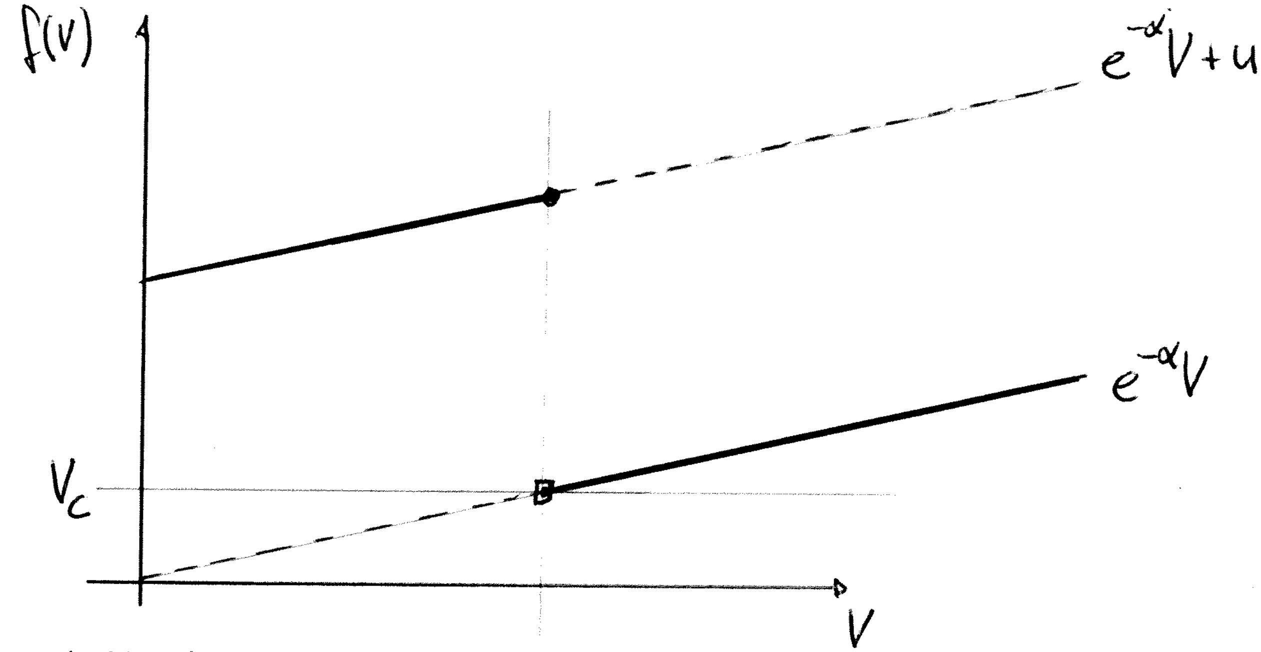
$V_{t+1} = \hat{V}_t + u$ if signal does come through and u is the increase in potential from signal.

Denote by V_c the critical threshold for $\hat{V}_t$, then

$$V_{t+1} = f(V_t) = \begin{cases} e^{-\alpha} V_t & \text{if } e^{-\alpha} V_t > V_c \\ e^{-\alpha} V_t + u & \text{if } e^{-\alpha} V_t \le V_c \end{cases}$$

a piecewise defined function !

Let's draw the updating function:

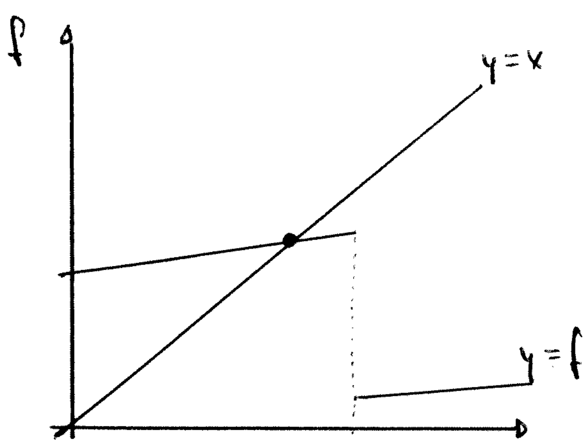


Now let's look at possible cases.

Fixed point: $f(V^*) = V^* \Rightarrow e^{-\alpha}V^* + u = V^* \Rightarrow V^* = \dfrac{u}{1-e^{-\alpha}}$

<u>if</u> $e^{-\alpha}V^* < V_c$

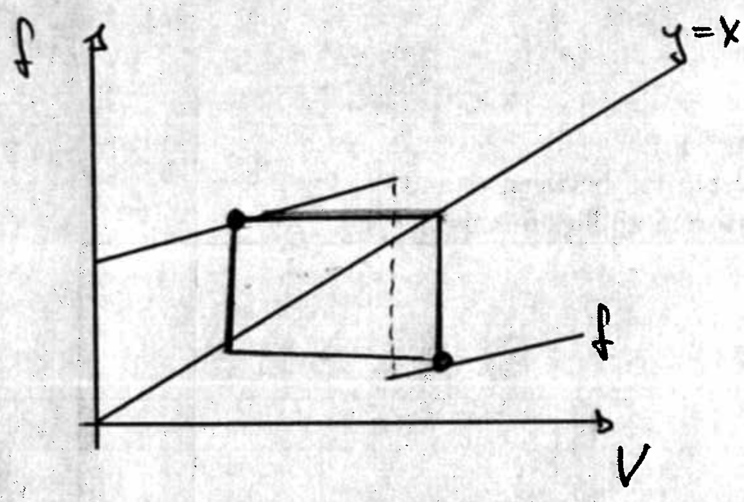
<u>Cases:</u> 1



If there is a fixed point, it is stable. E.g. $\alpha = 2$
$u = 1$
$V_c = 0.2$

The heart beats with every signal. Good

Case 2



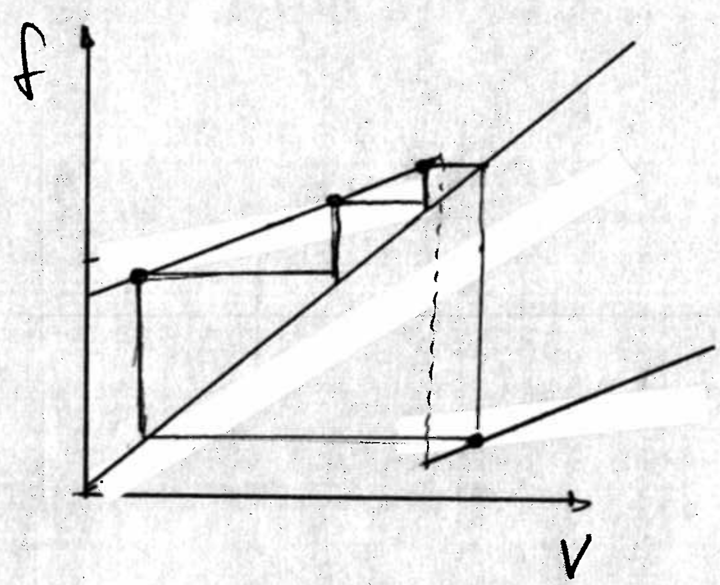
E.g. $\alpha = 1$

$u = 1$

$V_c = 0.2$

No fixed point. Solutions jump back and forth from the upper to the lower branch of f. $\Rightarrow$ Heart beats only every other signal (upper branch of f).

Case 3



No fixed point.

Solutions hit the top branch 3 times, then the bottom once.

$\Rightarrow$ Heart skips every 4^{th} beat

Try out
heartbeat.xls

E.g. $\left.\begin{array}{l}\alpha = 1.67 \\ u = 1 \\ V_c = 0.2\end{array}\right\}$ skips every 3^{rd}

E.g. $\left.\begin{array}{l}\alpha = 0.65 \\ u = 1 \\ V_c = 1\end{array}\right\}$ skips every 4^{th}

7.2 Limits, infinity and continuity – lecture

GOAL: Extend the material from the previous class to infinity; introduce continuity.

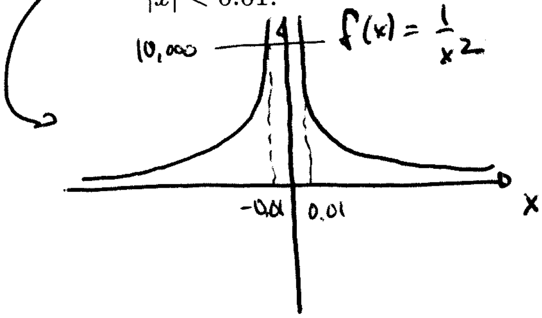
Definition: Infinite limits. We say that the limit of a function f as x approaches a is infinity (or negative infinity) if we can make $f(x)$ as large (negative and large) as we wish by choosing x very close to a. We write

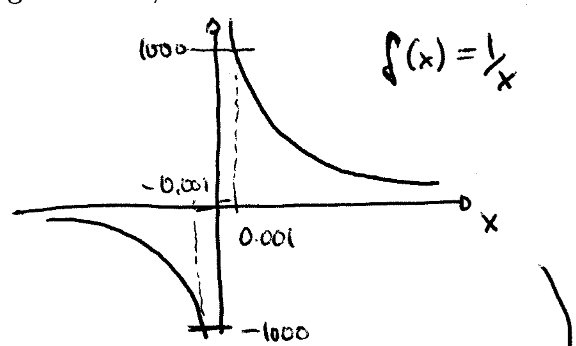
$$\lim_{x \to a} f(x) = \infty \qquad \text{or } \lim_{x \to a} f(x) = -\infty.$$

This definition applies also to one-sided limits. Geometrically, an infinite limit at a finite value a corresponds to a *vertical asymptote* of the graph at $x = a$.

Examples:

1. $f(x) = \frac{1}{x^2}$ and $a = 0$. The function is not defined at 0, and as $x \to 0$ we have a fixed positive number divided by smaller and smaller numbers. As a result, the limit is infinite. For example, if we want to make $f(x)$ larger than 10,000 then we have to make $|x| < 0.01$.



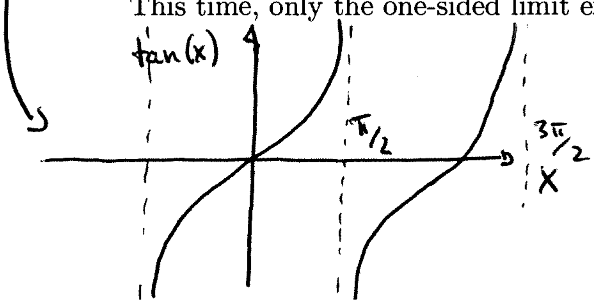


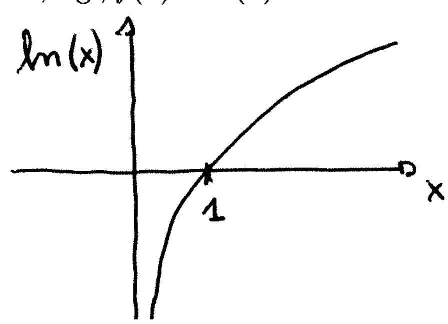
2. $f(x) = \frac{1}{x}$ and $a = 0$. Again, the function is not defined at 0, but we need to be careful with signs. When $x > 0$ the fraction is positive and gets arbitrarily large, when $x < 0$ it is negative and large in absolute value. Hence we have one-sided limits only. For example, if we want to make $f(x)$ larger than 1000, then we have to have $0 < x < 0.001$. To make $f(x) < 1000$ we have to choose $-0.001 < x < 0$.

3. All rational functions work similarly. When the denominator approaches zero and the numerator does not, then we need to check signs to see whether and from which side the function approaches $\pm\infty$.

4. Many more functions have similar properties, for example $f(x) = \tan(x) = \frac{\sin(x)}{\cos(x)}$ is a fraction of functions. The zeros of the denominator are not zeros of the numerator, hence there are infinite limits at those places.

5. Another class of functions with infinite limits are logarithms, e.g., $f(x) = \ln(x)$ as $x \to 0$. This time, only the one-sided limit exists.





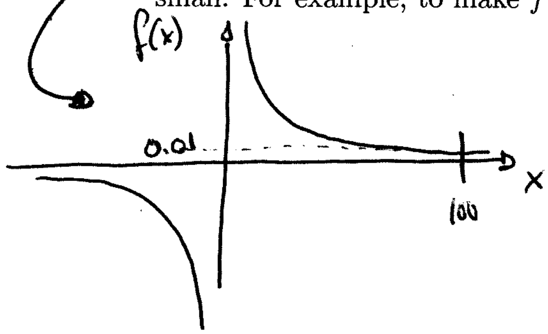
Definition: Limits at infinity. We say that the limit of a function f as x approaches ∞ equals L and write

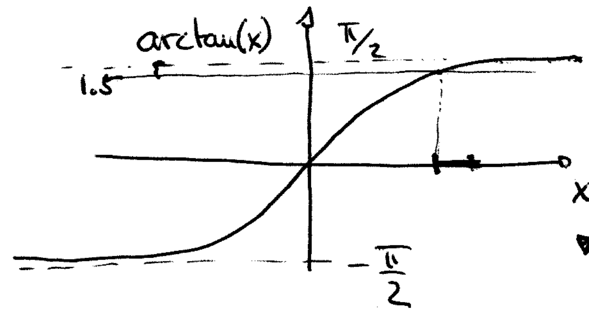
$$\lim_{x\to\infty} f(x) = L$$

if we can make $f(x)$ as close to L as we wish by choosing x arbitrarily large. We can also consider the limit $x \to -\infty$, but obviously, these limit can only be one-sided. A limit at infinity corresponds to a *horizontal asymptote* of the graph at $y = L$.

Examples:

1. The fraction $f(x) = 1/x$ is an important example. As $x \pm \infty$, the fraction approaches zero. More generally, if $g(x)$ can grow arbitrarily large, then $1/g(x)$ will grow arbitrarily small. For example, to make $f(x)$ smaller than 0.01, we have to choose $x > 100$.





2. Given a rational function, we divide numerator and denominator by the highest power of x, the we see what happens. For example

$$\lim_{x\to\infty} \frac{ax}{b+x} = \lim_{x\to\infty} \frac{\frac{ax}{x}}{\frac{b+x}{x}} = \lim_{x\to\infty} \frac{a}{b/x+1} = a$$

and

$$\lim_{x\to\infty} \frac{ax}{b+x^2} = \lim_{x\to\infty} \frac{\frac{ax}{x^2}}{\frac{b+x^2}{x^2}} = \lim_{x\to\infty} \frac{a/x}{b/x^2+1} = 0.$$

Note that for rational functions $r(x)$, if $\lim_{x\to\infty} r(x) = L$ then also $\lim_{x\to-\infty} r(x) = L$.

3. $f(x) = e^{ax}$ with $a > 0$ has $\lim_{x\to\infty} f(x) = \infty$ but $\lim_{x\to-\infty} f(x) = 0$.

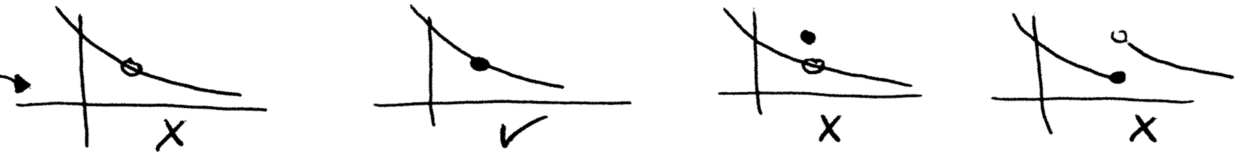
4. Recall the arctan function, the inverse function of tan? Its horizontal asymptotes are $\pm\pi/2$. Why? What are the vertical asymptotes of tan? And how are a function and its inverse related?

5. We can build fractions of functions other than polynomials, and those functions may have different limits as $x \to \pm\infty$. The idea of dividing by the "highest power" still works. For example

$$\lim_{x\to\infty} \frac{e^x-1}{5+2e^x} = \lim_{x\to\infty} \frac{\frac{e^x-1}{e^x}}{\frac{5+2e^x}{e^x}} = \lim_{x\to\infty} \frac{1-e^{-x}}{5e^{-x}+2} = 1/2.$$

On the other hand, $\lim_{x\to-\infty} \frac{e^x-1}{5+2e^x} = -1/5$.

The fastest growing function. More about that later



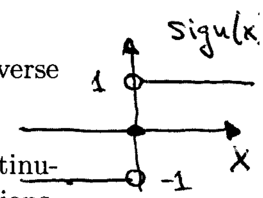
We have seen that the direct substitution rule often allows us to evaluate limits fairly easily and that many functions allow us to use this rule. We now give these functions a special name: continuous functions.

Definition: Continuity. A function f is called *continuous at a point* a if $f(a)$ exists, and if the limit $\lim_{x\to a} f(x) = f(a)$. In particular, this limit needs to exist. A function is called *continuous on an interval* if it is continuous at every point in that interval. Roughly, a function is continuous if it does not jump. If a function is continuous everywhere where it is defined, we say that it is continuous on its domain. If the context is clear, we simply say continuous.

Note: If you know that a certain function is continuous, then limits (within the domain of the function) are easily evaluated by the direct substitution rule.

Examples:

1. The following functions are continuous: constant, linear, exponential, logarithm, root, absolute value, sin, cos.
2. If f, g are continuous, then so are $f \pm g$, $f \cdot g$ f/g (provided $g \neq 0$) and $f \circ g$.
3. Therefore, all polynomials, rational functions (where defined), trigonometric and inverse trigonometric functions (where defined) are continuous.
4. The sign function is not continuous at $x = 0$. (There is nothing wrong about discontinuous functions. Many processes in real life are best desctibed by discontinuous functions. For example: harvesting, drug levels in body...)



Note: Look out for when checking continuity: division by zero, log or zero, piecewise defined functions at places where the function definition changes.

Example: Find c such that the following function is continuous at $x = 0$.

$$f(x) = \begin{cases} x + c, & x < 0 \\ \cos(x), & x \geq 0. \end{cases}$$

Theorem: Continuity and exchanging limits. If f is continuous and g is a function so that $\lim_{x\to a} g(x) = b$ exists, then we can take the limit of the composition

$$\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x)) = f(b).$$

Example: Find the limit $\lim_{x\to\infty} e^{1/x}$.

$$\lim_{x\to\infty} e^{1/x} = e^{\lim_{x\to\infty} 1/x} = e^0 = 1$$

Since the exponential function is continuous

Example:

$$\lim_{x\to 3} \left(\cos\left(\frac{7x+5}{(4x+2)^2}\right)\right) = \cos\left(\lim_{x\to 3} \frac{7x+5}{(4x+2)^2}\right) = \cos\left(\frac{26}{196}\right)$$

8.2 Differentiability - lecture

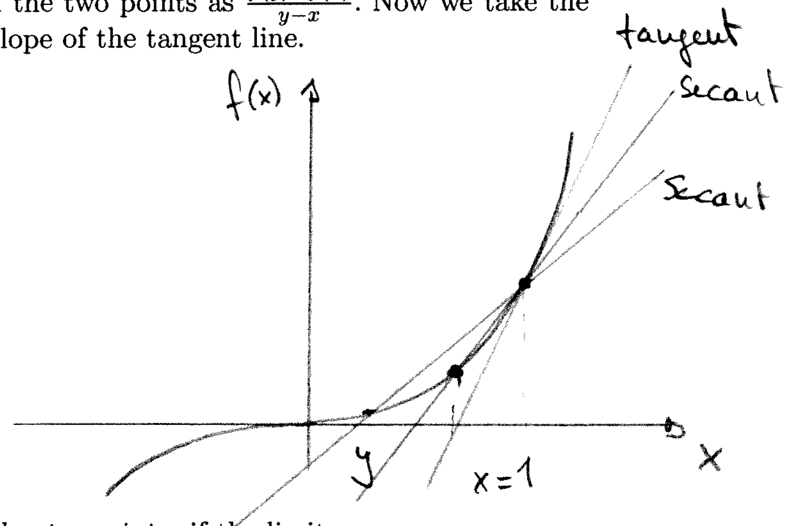
GOAL: Define the slope of a function at a point.

Idea: We know what the slope of a line is. Given two points, $x \neq y$ in the domain of function f, we write the slope of the secant line though the two points as $\frac{f(y)-f(x)}{y-x}$. Now we take the limit as $y \to x$ and define this limit to be the slope of the tangent line.

Example: If $f(x) = x^3$ and $x = 1$, then the slope of the secant line between y and x is $\frac{y^3-x^3}{y-x} = \frac{y^3-1}{y-1}$. If we now take the limit as $y \to 1$, we get

$$\lim_{y\to 1} \frac{y^3-1}{y-1} = \lim_{y\to 1}(y^2+y+1) = 3,$$

by simplifying, using the limit laws and direct substitution. Hence, the slope of the function $f(x) = x^3$ at $x = 1$ equals 3.



Definition: A function f is called *differentiable at a point* x if the limit

$$f'(x) := \lim_{y\to x} \frac{f(y)-f(x)}{y-x} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

exists. (Note that f has to be defined and continuous at x.) A function is called differentiable on an open interval, if it is differentiable at each point of the interval. We also write df/dx for the derivative.

Example: When f is a straight line, then this definition of a slope should agree with the slope of the function as we know it. The calculation below shows that this is indeed the case. Let $f(x) = mx + b$ then

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{m(x+h)+b-(mx+b)}{h} = \lim_{h\to 0} \frac{mh}{h} = m.$$

Example: For a more complicated example, let's choose $f(x) = \sqrt{x}$ and let's calculate the derivative in general for any $x > 0$ and not for a particular value of x. According to the definition

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h}.$$

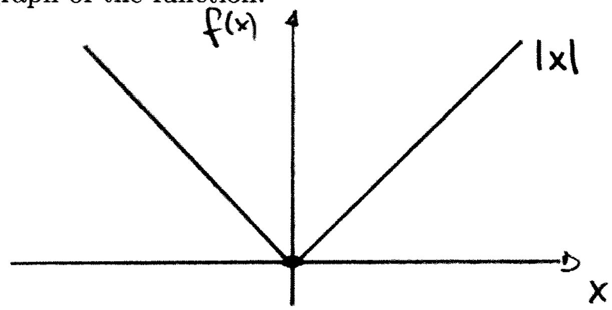
Now we rationalize the numerator and continue.

$$f'(x) = \lim_{h\to 0} \frac{(\sqrt{x+h}-\sqrt{x})(\sqrt{x+h}+\sqrt{x})}{h(\sqrt{x+h}+\sqrt{x})} = \lim_{h\to 0} \frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})} = \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{2\sqrt{x}},$$

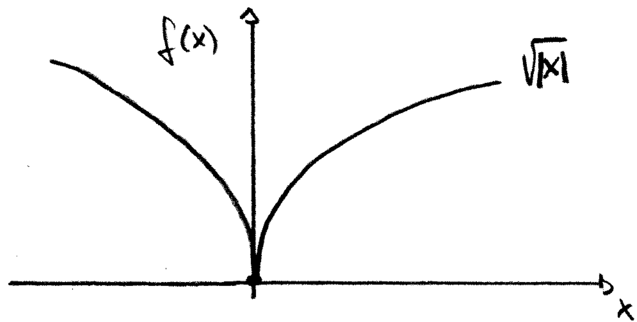
where the last equality results from direct substitution.

How a function can fail to be differentiable: There are many ways in which a function may fail to be differentiable at a point, corresponding to the ways in which a limit may fail to exist. The following examples stand for various ways in which the failure to be differentiable at a point corresponds to properties of the graph of the function.

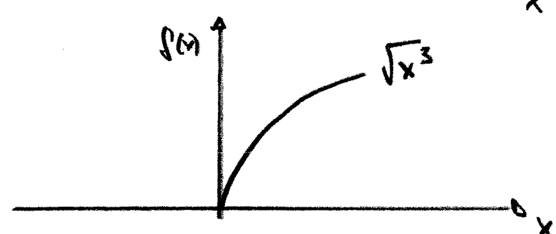
1. If the left and right limits exist and are finite but different, then the slope of the function is different, depending on the side form which one approaches the point. We have a corner. An example is $f(x) = |x|$ at $x = 0$. Note that $f(0) = 0$ is defined.



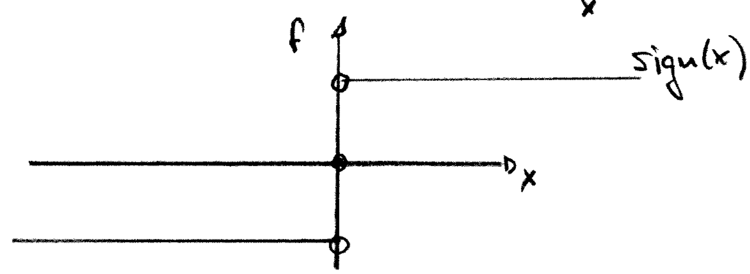
2. If the limit is infinite, then the derivative does not exist. A tangent line must be vertical, and we have a cusp. An example function is $f(x) = \sqrt{|x|}$ at $x = 0$. Note that $f(0) = 0$ is defined.



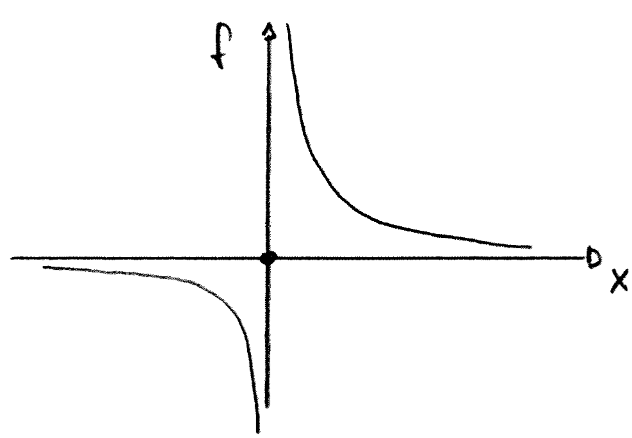
3. If only one limit (left or right) exists, then the derivative does not exist. For example, the function $f(x) = \sqrt{x^3}$, which is defined only for $x \geq 0$.



4. If the function is discontinuous at x then it cannot be differentiable at x. Example, the function $f(x) = \text{sign}(x)$ at $x = 0$.



5. If the function has a vertical asymptote, then the function is not differentiable there. Example function $f(x) = 1/x$ at $x = 0$.



Example: Let's do one more abstract example to practice the definition of derivative. Let's find the derivative of a general quadratic polynomial $f(x) = ax^2 + bx + c$.

$$f(x+h) - f(x) = a(x^2+2xh+h^2) + b(x+h)+c - [ax^2+bx+c]$$
$$= 2axh + ah^2 + bh = h(2ax + ah + b)$$

So $\lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0} \dfrac{h(2ax+ah+b)}{h} = \lim\limits_{h\to 0} (2ax+ah+b) = 2ax+b$

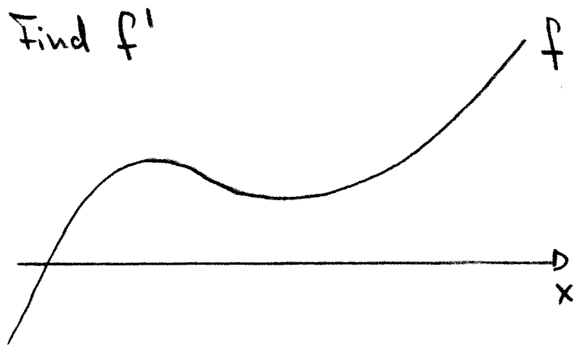
Derivatives and Graphs: Since we defined derivatives as slopes of tangent lines, there is a nice correspondence between the properties of the derivative and the graph of a function.

$$f'(x) > 0 \Leftrightarrow f \text{ increasing at } x$$
$$f'(x) = 0 \Leftrightarrow f \text{ has horizontal tangent at } x$$
$$f'(x) < 0 \Leftrightarrow f \text{ decreasing at } x$$

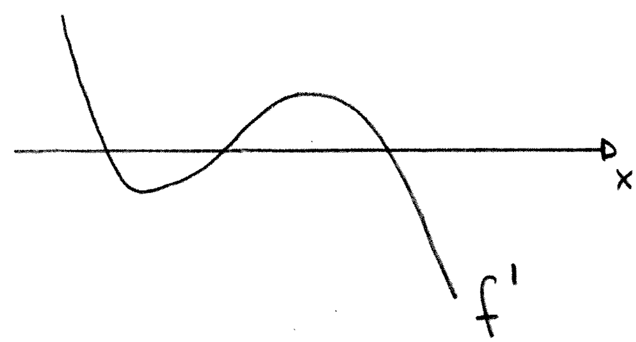
Definition: x is called a *critical point of* f if x is in the domain of f and either $f'(x) = 0$ or $f'(x)$ is not defined.

Example: We can find qualitative aspects of a derivative of a function by looking at the graph - and vice versa.

Find f'



Find f



Note: Calculation of derivative from the definition/from first principles can be done in simple cases, see above. But in general, this is a tedious and often tricky way of doing things. Instead, we fill find general rules that we can use to calculate derivatives faster. These rules have two aspects: there are rules of how derivatives behave with respect to function operations (adding, subtracting, multiplying, dividing, composing functions) and there are rules for certain classes of functions (polynomials, exponentials, logarithms, trigonometric functions). We start with some basic rules.

Differentiation rules: If the derivatives exist, then the following rules hold

1. If $f(x) = x^n$, then $f'(x) = nx^{n-1}$ The proof of that uses Binomial coefficients/Pascal's triangle.
2. $(f \pm g)'(x) = f'(x) \pm g'(x)$
3. If $h(x) = f(x) \cdot g(x)$ then $h'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$
4. If $h(x) = \frac{u(x)}{v(x)}$ then $h'(x) = \frac{u'(x)v(x) - u(x)v'(x)}{v^2(x)}$

Note: Even if you have seen all these rules in high school, I strongly recommend that you practice at least 5 differentiation exercises from the book every day for a week.

Why is the first rule true? $f(x) = x^n$ gives $f'(x) = nx^{n-1}$

Look back at the example on page 34: $f(x) = ax^2 + bx + c$

It gives all the important steps. We calculate

$$f(x+h) - f(x) = (x+h)^n - x^n$$

Then we need to know: $(x+h)^n = x^n + nx^{n-1}h + \underbrace{\ldots}_{\text{each term has factor } h^2 \text{ or more.}} + h^n$

So now $f(x+h) - f(x) = nx^{n-1}h + \underbrace{\ldots}_{\text{each term has factor } h^2 \text{ or more}} + h^n$

Then: $$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{nx^{n-1}h + \ldots + h^n}{h}$$

$$= \lim_{h \to 0} \Big(nx^{n-1} + \underbrace{\ldots}_{\text{each term has factor } h} + h^{n-1} \Big) = nx^{n-1}$$

direct substitution

Example: $f(x) = x^{2015} \to f'(x) = 2015\, x^{2014}$

But the rule also holds for non-integer exponents.

Example: $f(x) = \sqrt{x} = x^{\frac{1}{2}}$ so $n = \frac{1}{2}$

$$f'(x) = \frac{1}{2} x^{1-\frac{1}{2}} = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$ (compare bottom of page 32)

Example: $f(x) = \frac{1}{x^2} = x^{-2}$ so $n = -2$

$$f'(x) = -2x^{-3} = -\frac{2}{x^3}$$

The second rule tells us that we can differentiate sums simply by differentiating each term.

Example: $f(x) = x^5 + \frac{1}{x}$ then (with $n = 5$ and $n = -1$)

$$f'(x) = 5x^4 - \frac{1}{x^2}$$

And with a third rule we can differentiate even polynomials:

$$\left(c f(x)\right)' = c f'(x)$$

This rule is a special case of the product rule.

Example: $f(x) = 12\, x^{2015} + 7\, x^{10} - \frac{1}{5} x^5$

$$f'(x) = 12 \cdot 2015 \cdot x^{2014} + 70\, x^9 - x^4$$

Differentiation: part II

Example: We can interpret the derivative as the slope of the tangent line. But there are many examples in physics, where derivatives have physical meaning. For example: derivative of location is velocity; derivative of velocity is acceleration.

The position of a falling rock, released 50 m above ground, is given by

$$p(t) = 50 - \frac{a}{2}t^2 \qquad a: \text{acceleration by gravity}$$

What is the terminal velocity? i.e. how fast is the rock when it hits the ground.

1) Find t when the rock hits the ground: $p(t) = 0 \Leftrightarrow t = \sqrt{\frac{100}{a}}$

2) Find the derivative: $p'(t) = -at$

3) Substitute terminal time: $p'\left(\sqrt{\frac{100}{a}}\right) = -a\sqrt{\frac{100}{a}} = -\sqrt{100a}$

On earth: $a \simeq 9.81\ m/s^2 \approx 10\ m/s^2$ so $p'(\sqrt{10^3}) = -31.6\ \frac{m}{s}$ (why <0?)

On the moon: $a \approx 1.62\ m/s^2$ ... $p'(\ ) \approx -12.72\ \frac{m}{s}$

On Jupiter: $a \simeq 22.88\ m/s^2$ ... $p'(\ ) \approx -47.83\ \frac{m}{s}$

Check the units!

Okay. So last class, we had the definition of the derivative as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

We also saw the product rule

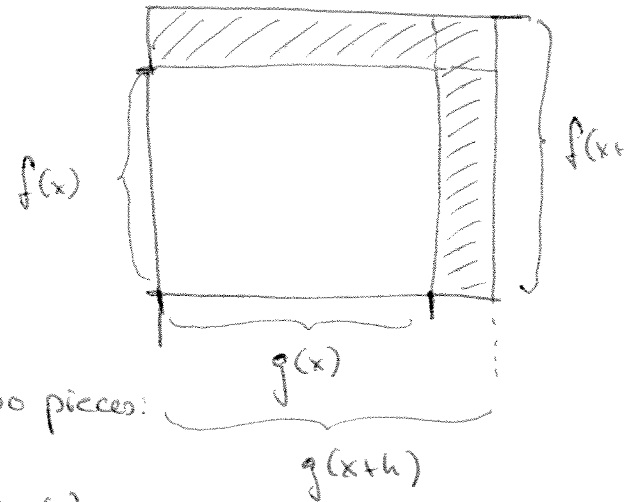
$$(fg)' = f'g + fg'$$

How do we know that the product rule is true? Can we get it from first principles?

The product rule

$$\lim_{h\to 0} \frac{1}{h}\left[f(x+h)g(x+h) - f(x)g(x)\right] = ?$$

Let's look at the picture, where the product is interpreted as an area: $A = f(x)\cdot g(x)$.

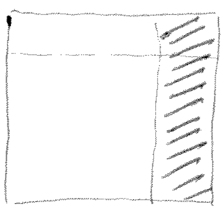


Then the difference

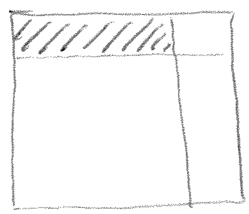
$f(x+h)g(x+h) - f(x)g(x)$ is the /// area.

We can break this shaded area down into two pieces:

$$\underbrace{f(x+h)g(x+h) - f(x+h)g(x)} + \underbrace{f(x+h)g(x) - f(x)g(x)}$$



$+$



Put this expression into the formula for the derivative:

$$\lim_{h\to 0} \frac{1}{h}\left[f(x+h)g(x+h) - f(x)g(x)\right] = \lim_{h\to 0} \frac{1}{h}\Big[f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)\Big]$$

limit laws

$$= \lim_{h\to 0} \frac{1}{h}\left[f(x+h)g(x+h) - f(x+h)g(x)\right] + \lim_{h\to 0} \frac{1}{h}\left[f(x+h)g(x) - f(x)g(x)\right]$$

limit laws

$$= \lim_{h\to 0} f(x+h)\frac{g(x+h)-g(x)}{h} + \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}\, g(x)$$

laws and continuity

$$= f(x)\,g'(x) + f'(x)\,g(x)$$

Example: Calculate the derivative of

$$A(x) = (2x+3)(5x-2)$$

in two different ways.
1) Product rule
2) Multiply and power rule.

Derivatives of exponential functions

Now let's get to a new class of functions: exponential functions. $f(x)=e^x$

To find the derivative, let's start with the only thing we know: the definition:

$$\lim_{h\to 0} \frac{1}{h}\left[e^{x+h} - e^x\right] = \lim_{h\to 0} \frac{1}{h}\left[e^x e^h - e^x\right] = \lim_{h\to 0} e^x \frac{e^h - 1}{h} \underset{\text{limit properties}}{=} e^x \lim_{h\to 0} \frac{e^h-1}{h}$$

This is not quite as good as we would like it: we don't have a way to evaluate the limit. However, it is also a great result: It tells us that we need to calculate only one limit $\left(\lim_{h\to 0}\frac{e^h-1}{h}\right)$ to get the derivative for <u>any</u> x. Use your calculator to guess the limit.

h	0.1	0.01	0.001	−0.1	−0.01	−0.001
$\frac{e^h-1}{h}$						

Hence, we find $\frac{d}{dx} e^x = e^x$. In fact $f(x)=e^x$ is the <u>only</u> function that satisfied $f'=f$ (aside from multiples $f(x) = Ke^x$.)

Examples: 1) $f(x) = x^2 e^x$ find $f'(x)$

2) $f(x) = \dfrac{e^x}{\sqrt{x}}$ find $f'(x)$

3) $f(x) = \dfrac{x^{1/3}}{e^x}$ find $f'(x)$

How about $f(x) = e^{4x}$ find $f'(x)$?

This can be done similarly to the above (with a little trick), but it is also an example for a general question:

Suppose $F(x) = f(g(x))$, how do you find $F'(x)$?

Example: 1) $g(x) = 4x$, $f(y) = e^{y} \to F(x) = f(g(x)) = e^{4x}$

2) $g(x) = x^2+3x-2$, $f(y) = \sqrt{y} \to F(x) = f(g(x)) = \sqrt{x^2+3x-2}$

3) $g(x) = 1+x^2$, $f(y) = \frac{1}{y} \to F(x) = f(g(x)) = \frac{1}{1+x^2}$

Remember:

$$\begin{tikzcd} x \arrow[r, "g"] \arrow[rr, bend right, "F(x)"'] & g(x) \arrow[r, "f"] & f(g(x)) \end{tikzcd}$$

The rule is: $\boxed{F'(x) = f'(g(x)) \cdot g'(x)}$

To see why the rule is true, we start with

$$f'(y) = \lim_{h\to 0} \frac{f(y+h) - f(y)}{h}$$

now we replace h by $g(x+\ell) - g(x)$ as $\ell \to 0$ and y by $g(x)$

$$= \lim_{\ell\to 0} \frac{f(g(x)+g(x+\ell)-g(x)) - f(g(x))}{g(x+\ell)-g(x)}$$

$$= \lim_{\ell\to 0} \frac{f(g(x+\ell))-f(g(x))}{\ell} \quad \frac{\ell}{g(x+\ell)-g(x)}$$

$$= (f\circ g)'(x) \cdot \frac{1}{g'(x)}.$$ Now multiply over to get

$$f'(g(x)) \cdot g'(x) = (f\circ g)'(x).$$

Example: $F(x) = e^{4x}$: $f(y) = e^y$, $f'(y) = e^y$

$g(x) = 4x$, $g'(x) = 4$

$$\Rightarrow F'(x) = f'(g(x)) \cdot g'(x) = e^{4x} \cdot 4$$

Example: $F(x) = \sqrt{x^2+3x+2}$: $f(y) = \sqrt{y}$, $f'(y) = \frac{1}{2\sqrt{y}}$

$g(x) = x^2+3x+2$, $g'(x) = 2x+3$

$$\Rightarrow F'(x) = f'(g(x))\, g'(x) = \frac{1}{2\sqrt{x^2+3x+2}} \cdot (2x+3)$$

Example: $F(x) = e^{-x^2}$: $f(y) = e^y$, $f'(y) = e^y$

$g(x) = -x^2$, $g'(x) = -2x$

$$\Rightarrow F'(x) = f'(g(x))\, g'(x) = e^{-x^2} \cdot (-2x).$$

Gaussian Distribution
"Bell curve"

Example: $F(x) = x^n e^{-x}$: $f(y) = e^y$ $f'(y) = e^y$

$g(x) = -x$ $g'(x) = -1$

$$\Rightarrow F'(x) = n x^{n-1} e^{-x} - x^n e^{-x} = (n-x)\, x^{n-1} e^{-x}$$

Gamma Distribution

Application to differentiating inverse function:

Apply the chain rule to $(f \circ f^{-1})(x) = x$. This is by definition of f^{-1}.

Differentiate both sides: $f'(f^{-1}(x)) \cdot (f^{-1}(x))' = 1$

Solve for $\boxed{(f^{-1}(x))' = \frac{1}{f'(f^{-1}(x))}}$

Examples: 1) Derivative of $\ln(x)$.

$$e^{\ln x} = x \qquad \text{so } f(x) = e^x, \quad f^{-1}(x) = \ln(x)$$

$$f'(x) = e^x$$

$$\left(f^{-1}(x)\right)' = \frac{1}{f'(f^{-1}(x))} = \frac{1}{e^{\ln x}} = \frac{1}{x}$$

2) Derivative of $\sqrt{x}$.

With $f(x) = x^2$ and $f^{-1}(x) = \sqrt{x}$ and $f'(x) = 2x$, we get

$$\left(\sqrt{x}\right)^2 = x \qquad \text{for } x \geq 0$$

and $$\left(f^{-1}(x)\right)' = \frac{1}{f'(f^{-1}(x))} = \frac{1}{2\sqrt{x}}$$

Differentiation III: Sine, cosine and implicit.

One large class of functions whose derivatives we don't know yet are trigonometric functions. Let's start with $f(x) = \sin(x)$, and let's go back to the definition.

$$f'(x) = \lim_{h\to 0} \frac{1}{h}\Big(\sin(x+h) - \sin(x)\Big) = \lim_{h\to 0} \frac{1}{h}\Big(\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)\Big)$$

$$= \sin(x) \lim_{h\to 0} \frac{\cos(h)-1}{h} + \cos(x) \lim_{h\to 0} \frac{\sin(h)}{h}$$

Observation: When we found $\frac{d}{dx}e^x$, we needed to compute one limit only. Here, we need two.

With a calculator, find your best guess for

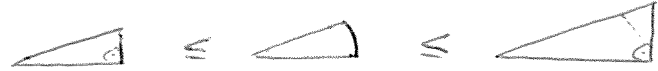
(a): $\displaystyle\lim_{h\to 0} \frac{\cos(h)-1}{h}$ (b): $\displaystyle\lim_{h\to 0} \frac{\sin(h)}{h}$

Now some geometry.

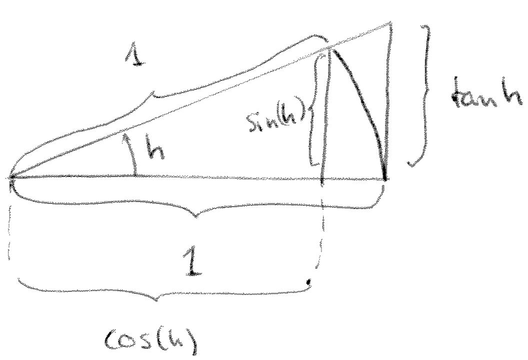
for (a) The expression is the derivative of $f(x) = \cos(x)$ at $x=0$. But since $\cos(x)$ is an even function, its derivative at zero should be zero.

for (b) Look at the triangles and sector of a circle.

The areas are



$$\frac{1}{2}\sin(h)\cos(h) \le \frac{1}{2}\cdot 1\cdot h \le \frac{1}{2}\cdot 1\cdot \tan(h)$$



$$\Rightarrow \quad \cos(h) \le \frac{h}{\sin(h)} \le \frac{1}{\cos(h)}$$

$$\Rightarrow \quad \frac{1}{\cos(h)} \le \frac{\sin(h)}{h} \le \cos(h)$$

Now take limits and remember: $\displaystyle\lim_{h\to 0} \cos(h) = 1$.

So, we have the rule $\frac{d}{dx}\sin(x) = \cos(x)$

How about $\frac{d}{dx}\cos(x)$? Be clever!

$$\frac{d}{dx}\cos(x) = \frac{d}{dx}\sin\left(x+\frac{\pi}{2}\right) = \cos\left(x+\frac{\pi}{2}\right) = \sin(x+\pi) = -\sin(x)$$

<u>Remember:</u>

$$\boxed{\begin{aligned}\sin(x)' &= \cos(x)\\ \cos(x)' &= -\sin(x)\end{aligned}}$$

And so, if we differentiated again?

$$\frac{d}{dx}\left(\frac{d}{dx}\sin(x)\right) = \frac{d}{dx}\cos(x) = -\sin(x)$$

→ The second derivative of the function equals the negative of the function.

[Almost as exciting as the rule for e^x.]

<u>Example:</u> 1) $f(t) = A + B\cos\left(\frac{2\pi}{T}(t-\phi)\right)$

$$f'(t) = -B\sin\left(\frac{2\pi}{T}(t-\phi)\right)\frac{2\pi}{T}$$

2) $f(x) = 3\sin(x^2)$

$$f'(x) = 3\cos(x^2)\cdot 2x$$

Note:

$\cos^2(x) = \cos(x)\cdot\cos(x)$

$\cos(x^2) = \cos(x\cdot x)$

3) $g(w) = 5\cos^2(\sqrt{x})$

$$g'(w) = 5\,2\cos(\sqrt{x})(-\sin\sqrt{x})\cdot\frac{1}{2\sqrt{x}}$$

4) $f(x) = \cos\left(\sqrt{x} + \sin(x^3)\right)$

$$f'(x) = \dots$$

Let's apply these new rules to differentiate more trigonometric functions

- $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$ use quotient rule

$$(\tan(x))' = \frac{\cos(x)\cos(x) - \sin(x)\cdot(-\sin(x))}{\cos^2(x)} = \frac{\cos^2(x)+\sin^2(x)}{\cos^2(x)} = \begin{cases} 1 + \tan^2(x) \\ \dfrac{1}{\cos^2(x)} = \sec^2(x) \end{cases}$$

And let's do an advanced application of differentiating inverse functions:

$\cos^{-1}(x) = \arccos(x)$. Note: $\cos^{-1}(x) \neq \dfrac{1}{\cos(x)}$

We have the relation: $\cos\left(\cos^{-1}(x)\right) = x$

Differentiate with the chain rule:

$$\left[\cos\left(\cos^{-1}(x)\right)\right]' \cdot \left(\cos^{-1}(x)\right)' = 1$$

Solve for $\cos^{-1}(x)'$:

$$\left(\cos^{-1}(x)\right)' = \frac{1}{\cos\left(\cos^{-1}(x)\right)'} = \frac{-1}{\sin\left(\cos^{-1}(x)\right)} = \frac{-1}{\sqrt{1-\cos^2\left(\cos^{-1}(x)\right)}}$$

Now use $\sin^2(x)+\cos^2(x) = 1$ to get $\sin(y) = \sqrt{1-\cos^2(y)}$

Finally use $\cos\left(\cos^{-1}(x)\right) = x$

$$= \frac{-1}{\sqrt{1-x^2}}.$$

So $\dfrac{d}{dx}\cos^{-1}(x) = \dfrac{-1}{\sqrt{1-x^2}}$

Exercise: Derive the formula for the derivative of $\arcsin(x) = \sin^{-1}(x)$.

Another application of the chain rule: implicit differentiation.

Example: The relation $x^2+y^2=1$ defines a circle of radius 1 in the plane. To find, say, a tangent line to the circle, we need to know the slope at a point.

1st approach: Solve for y and differentiate: $y = \pm\sqrt{1-x^2}$

$$y' = \pm\frac{-2x}{2\sqrt{1-x^2}}$$

Now, given a point x, we can find one of the two corresponding points y on the circle, evaluate the slope, and write down the tangent line equation.

Eg. $x=\frac{1}{2} \to y = \pm\sqrt{\frac{3}{4}} \to y' = \pm\frac{-\frac{1}{2}}{\sqrt{\frac{3}{4}}} = \mp\sqrt{\frac{1}{3}}$

Then the line is (for $y=+\sqrt{\frac{3}{4}}$)

$$y-\sqrt{\tfrac{3}{4}} = -\sqrt{\tfrac{1}{3}}\left(x-\tfrac{1}{2}\right)$$

2nd approach: If we don't want to (or cannot) solve for y, we write $y=y(x)$ anyway and differentiate the entire equation, using the chain rule

So $x^2+(y(x))^2 = 1 \to 2x + 2y(x)\,y'(x) = 0$

Solve for y': $y'(x) = \dfrac{-2x}{2y(x)} \underset{\uparrow}{=} \dfrac{-x}{\pm\sqrt{1-x^2}}$

$x^2+y^2=1$ from the equation.

$\Rightarrow$ We can find y' from a <u>linear</u> equation, even if we cannot express y in terms of x.

Example: Find $y'(x)$ where $e^x y^2 - 5y = 3x + \ln y$

There is no way to solve for y in terms of x. Yet we can differentiate implicitly.

$$e^x (y(x))^2 - 5y(x) = 3x + \ln y(x)$$

remember, we differentiate with respect to x.

$$e^x (y(x))^2 + e^x \cdot 2y(x) y'(x) - 5y'(x) = 3 + \frac{y'(x)}{y(x)}$$

Now solve for y':

$$\left(e^x \cdot 2y(x) - 5 - \frac{1}{y(x)}\right) y'(x) = 3 - e^x (y(x))^2$$

$$\Rightarrow \quad y'(x) = \frac{3 - e^x (y(x))^2}{2e^x y(x) - 5 - \frac{1}{y(x)}}$$

<u>Logarithmic differentiation</u>

Look at $f(x) = x^x$. Cannot use power rule, nor exponential.
But can take ln on both sides and differentiate both sides:

$$\ln f(x) = x \ln x \quad \rightarrow \quad \frac{f'(x)}{f(x)} = \ln x + x \cdot \frac{1}{x} = \ln x + 1$$

Solve for $f'(x) = f(x)(\ln x + 1) = x^x \cdot (\ln x + 1)$

Alternative: $f(x) = x^x = e^{x \ln x}$ then by the chain rule

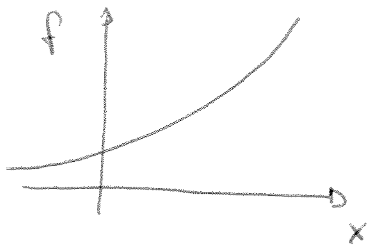
$$f'(x) = e^{x \ln x} \cdot (\ln x + 1) = x^x (\ln x + 1)$$

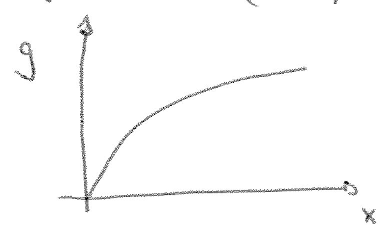
Second derivatives and curve sketching

Motivation: The derivative of a function tells us something about the slope of a function. For example, if $f'(x) > 0$ then f is increasing at x. However, there are different ways in which a function can be increasing. Look at these two:

$$f(x) = e^x \qquad\qquad g(x) = \frac{x}{1+x}, \quad x \geq 0$$

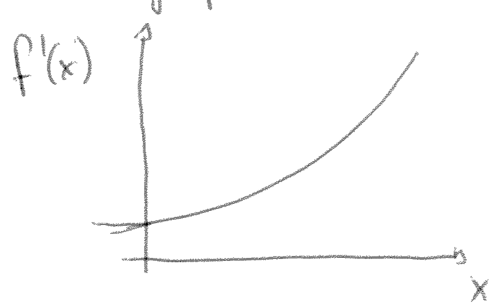
$$f'(x) = e^x > 0 \qquad\qquad g'(x) = \frac{1}{(1+x)^2} > 0$$

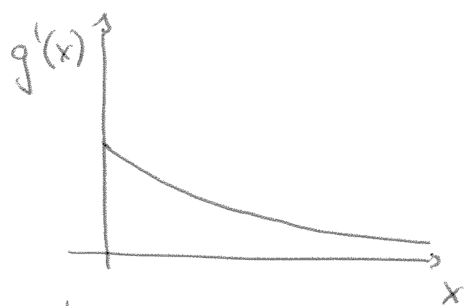




Both functions are increasing, but the rate of increase increases in f whereas the rate of increase decreases in g.

Let's graph the derivatives of f and g to see:





So f' is an increasing function and g' is a decreasing function.

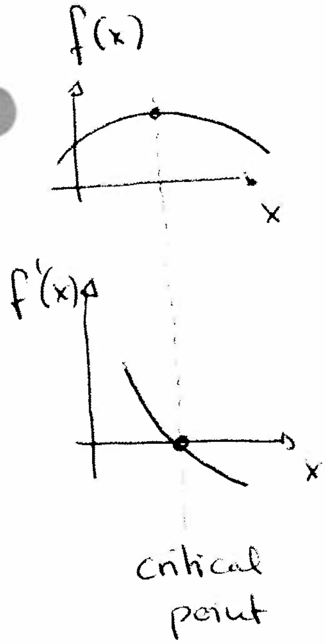
Let's try to formalize this via derivatives:

$$\frac{d}{dx}\left(f'(x)\right) = \frac{d}{dx} e^x = e^x > 0 \qquad \frac{d}{dx}\left(g'(x)\right) = \frac{d}{dx}\left(\frac{1}{1+x^2}\right) = \frac{-2}{(1+x)^3} < 0$$

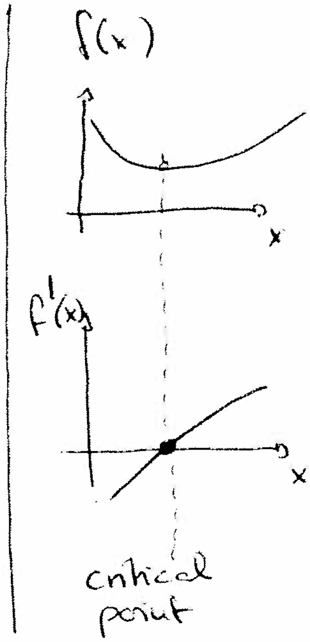
$x \geq 0$

$\Rightarrow$ Just like the (first) derivative tells us whether the function is increasing or decreasing the second derivative $f''(x) = \frac{d}{dx} f'(x) = \frac{d^2}{dx^2} f(x)$, tells us whether the derivative f' is increasing or decreasing.

Let's look at another picture



At this critical point, f goes from increasing to decreasing, so that f' goes from positive to negative. But then f' decreases and so $f'' < 0$

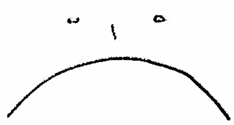


At this critical point, f goes from decreasing to increasing, so that f' goes from negative to positive. But that means that f' is increasing, and so $f'' > 0$

$\Downarrow$

$f'' < 0$

"sad face function"



called: <u>concave down</u>

$\Downarrow$

$f'' > 0$

"happy face function"

called <u>concave up</u>

<u>Example</u>:

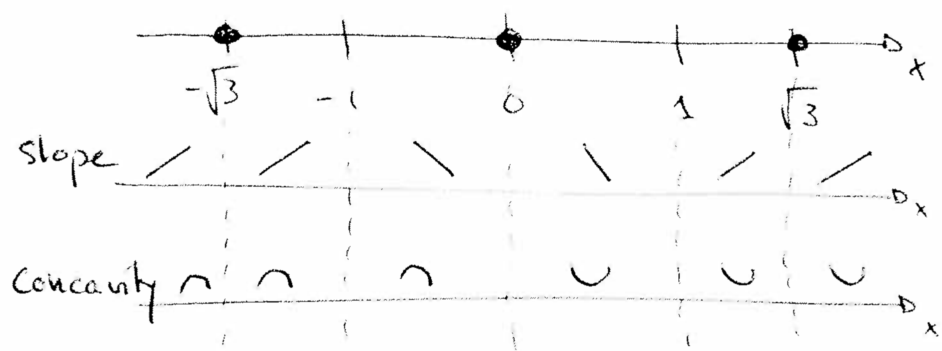
$$\left.\begin{aligned} f(x) &= x^3 - 3x \\ f'(x) &= 3x^2 - 3 \\ f''(x) &= 6x \end{aligned}\right\} \Rightarrow$$

$f(x) = 0$ for $x = 0$, $x = \pm\sqrt{3}$

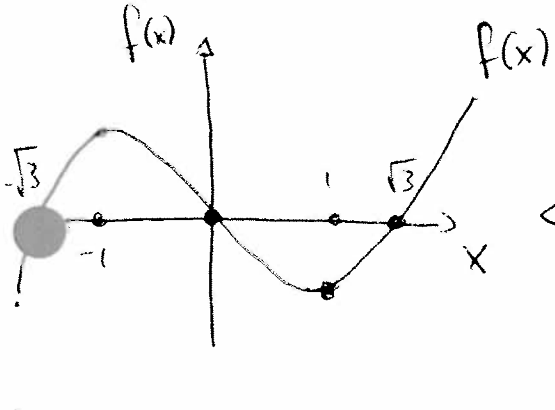
$f'(x) = 0$ for $x = \pm 1$ critical points

Concave up for $x > 0$, concave down for $x < 0$

$\Downarrow$



$\Leftarrow$



Summary: The second derivative is the derivative of the first derivative

$$f''(x) = \left(f'(x)\right)' = \frac{d}{dx}\left(\frac{d}{dx} f(x)\right) = \frac{d^2}{dx^2} f(x).$$

It gives us information about the curvature: concave up or down. The curvature may change if $f''(x) = 0$. If it does, then the point is called a point of inflection.

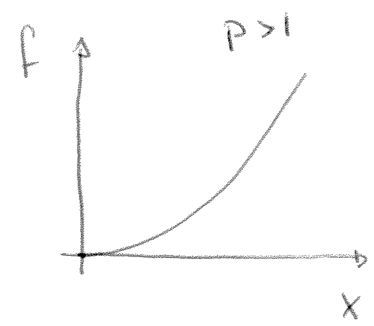
If, at a critical point x^*, so that $f'(x^*) = 0$, we have $f''(x^*) > 0$, then x^* is a local minimum, and if $f''(x^*) < 0$, then x^* is a local maximum.

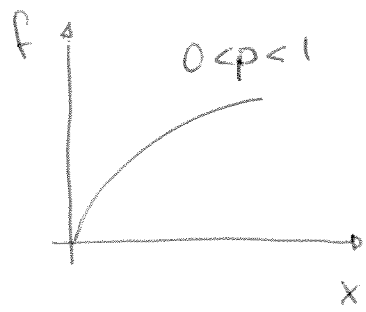
Examples: The power functions $f(x) = x^p$ for $x > 0$.

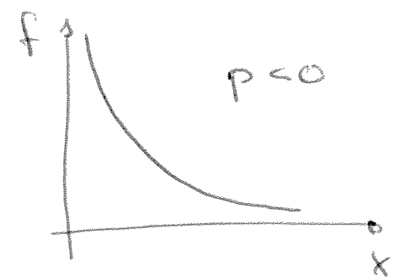
Their derivatives: $f'(p) = p x^{p-1}$, $f''(x) = p(p-1)x^{p-2}$

So

p	f'	f''
$p > 1$	+	+
$0 < p < 1$	+	−
$p < 0$	−	+



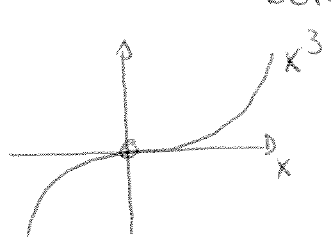




Note: Critical points and inflection points:

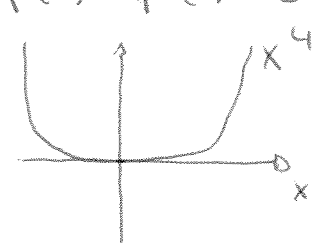
$f(x) = x^3$, $f'(x) = 3x^2$, $f''(x) = 6x$ | $f'(x) = x^4$, $f'(x) = 4x^3$, $f''(x) = 12x^2$

Both functions have $f'(0) = f''(0) = 0$



inflection ✓

max/min X



max/min ✓

inflection X

Applications of derivatives: curve sketching.

Given $f(x) = 1 + \frac{1}{x} - \frac{2}{x^2} = \frac{x^2+x-2}{x^2}$, sketch its graph.

1) Domain of f is $\mathbb{R}\setminus\{0\}$: all real numbers except zero.
$f(x) = 0$ for $x = 1$ and $x = -2$

2) As $x \to 0$, the numerator approaches $-2 < 0$ and the denominator approaches 0. Hence, by choosing x close to zero, we can make $f(x)$ as negative as we want, so $\lim_{x\to 0} f(x) = -\infty$

$\Rightarrow$ vertical asymptote at $x = 0$

3) As $x \to \infty$, we have $\lim_{x\to\infty} f(x) = 1$ since $\lim_{x\to\infty} \frac{1}{x} = 0$ and $\lim_{x\to\infty} \frac{2}{x^2} = 0$

Similarly for $x \to -\infty$: $\lim_{x\to-\infty} f(x) = 1$

$\Rightarrow$ horizontal asymptote at $y = 1$.

4) Derivative: $f'(x) = -\frac{1}{x^2} + \frac{4}{x^3} = \frac{4-x}{x^3}$

$f'(x) = 0$ when $x = 4$ and $f'(x)$ not defined when $x = 0$. But $x = 0$ is not in the domain of f. Hence, the critical point is $x = 4$.

Slope: If $x < 0$, then $x^3 < 0$ and $4 - x > 0$ so $f'(x) < 0$

If $0 < x < 4$, then $x^3 > 0$ and $4 - x > 0$ so $f'(x) > 0$

If $x > 4$, then $x^3 > 0$ and $4 - x < 0$ so $f'(x) < 0$

In particular, at $x = 4$, f goes from increasing to decreasing and therefore $x = 4$ is a local maximum.

1330 (42.5)

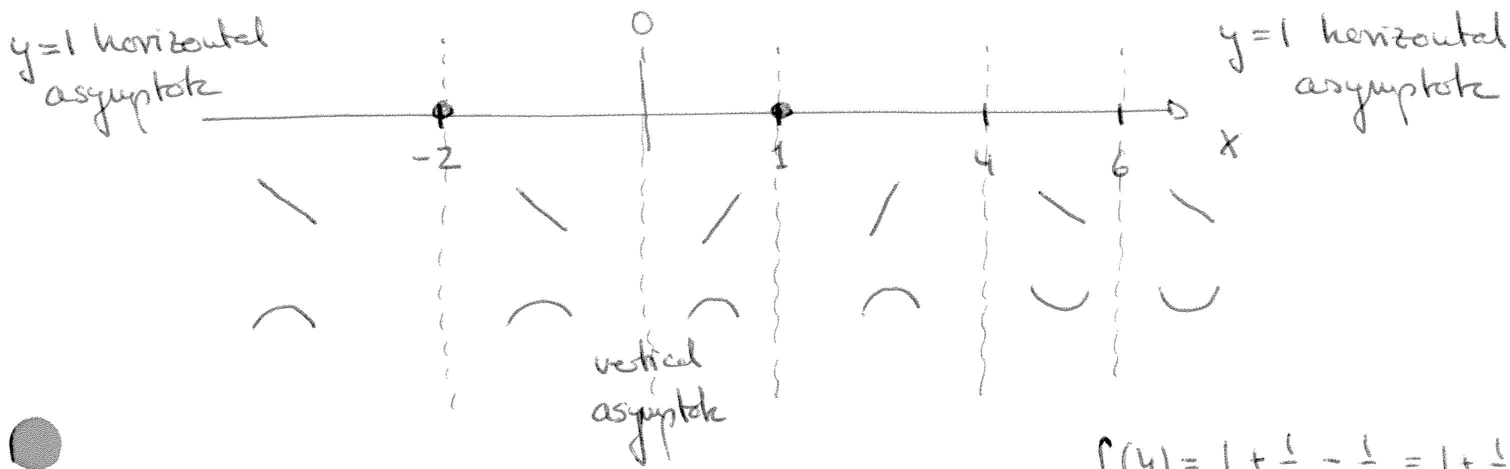
5) Second derivative: $f''(x) = \frac{2}{x^3} - \frac{12}{x^4} = \frac{2x-12}{x^4}$

$f''(x) = 0$ for $x = 6$ $\Rightarrow$ potential point of inflection.

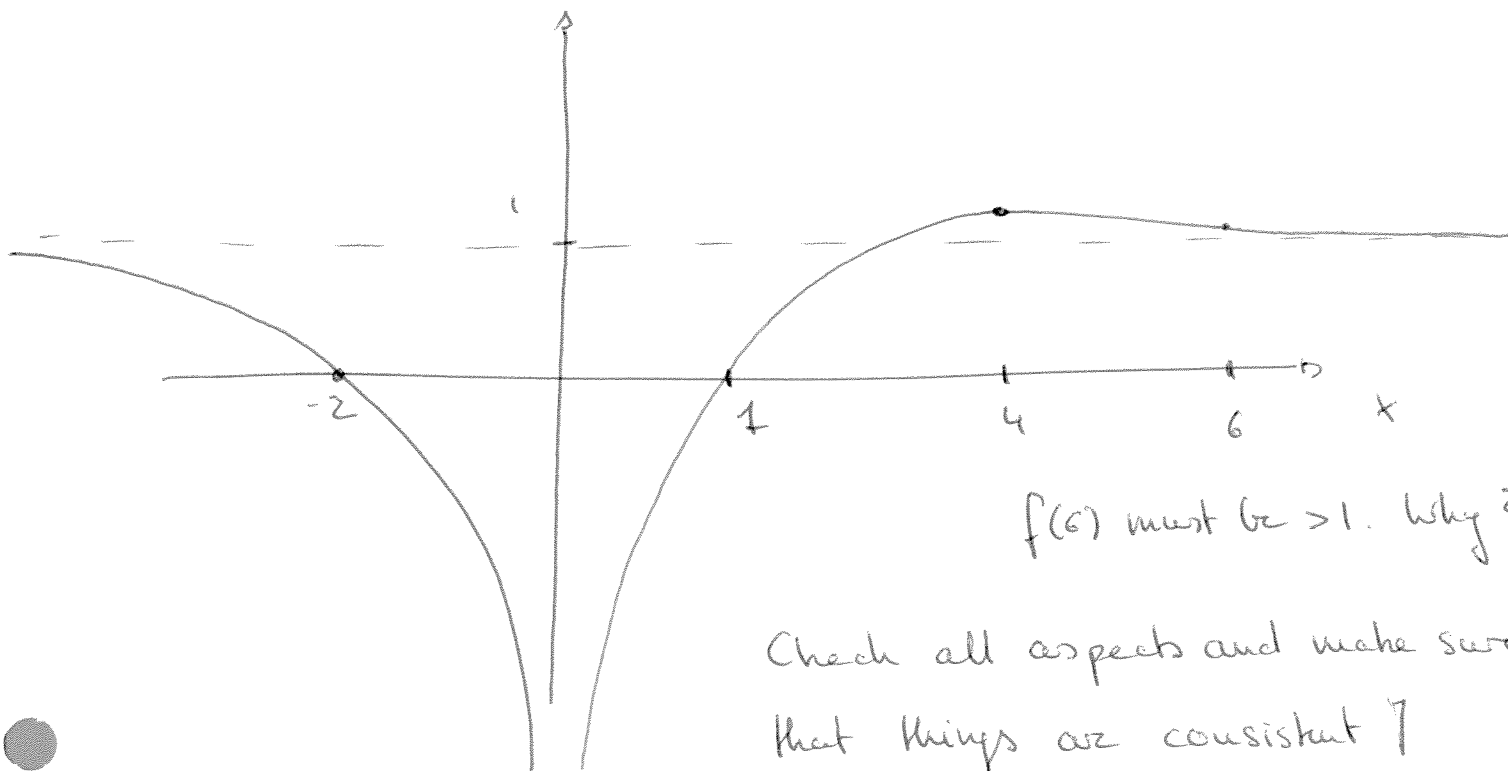
$f''(x) > 0$ if $2x > 12$ or $x > 6$ $\rightarrow$ concave up

$f''(x) < 0$ if $2x < 12$ or $x < 6$ $\rightarrow$ concave down.

$\Rightarrow x = 6$ <u>is</u> a point of inflection



$f(4) = 1 + \frac{1}{4} - \frac{1}{8} = 1 + \frac{1}{8} > 1$



$f(6)$ must be > 1. Why?

Check all aspects and make sure that things are consistent!

Extreme values

Goal: We want to know more about extreme values (maxima and minima) of functions. We saw something about them last class when curve sketching, and we will see lots more next class, when we ask optimization problems. Here, we want to characterize the different kinds of extrema that there are, and we want to find ways to calculate / identify them. We start with some terminology and end with a very general and powerful theorem.

Definition: An absolute (or global) maximum of a function f occurs at point c in the domain of f if $f(x) \leq f(c)$ for all $x \in \mathrm{Dom}(f)$. Similarly, for minimum we require $f(x) \geq f(c)$ for all $x \in \mathrm{Dom}(f)$.

Example: 1) $f(x) = x^2$ has a global minimum at $c=0$ where $f(0)=0$, Since $f(x)=x^2>0$ for all $x \neq 0$. Has no global maximum.

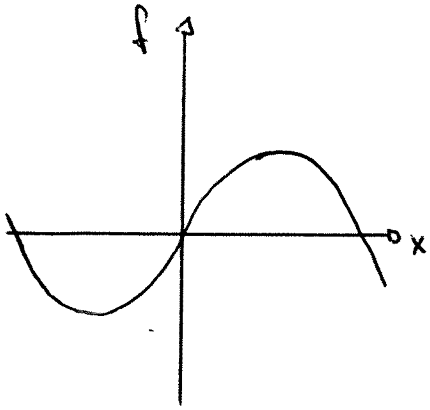
2) $f(x) = \sin(x)$ has a global maximum at $c = \frac{\pi}{2}$ where $f(\frac{\pi}{2}) = \sin(\frac{\pi}{2}) = 1$. It also has a global minimum at $c=-\frac{\pi}{2}$. In fact, the global maximum of 1 arises infinitely often, as does the global minimum.

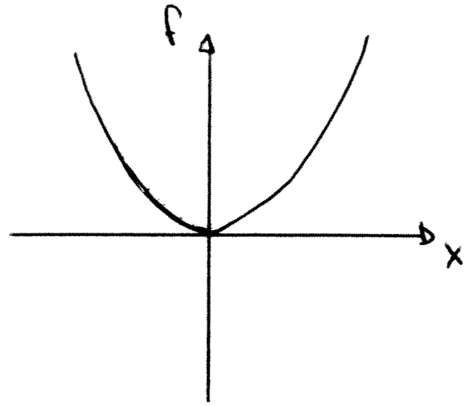
3) $f(x) = x^3$ has no global maximum and no global minimum. The function is unbounded in both directions.

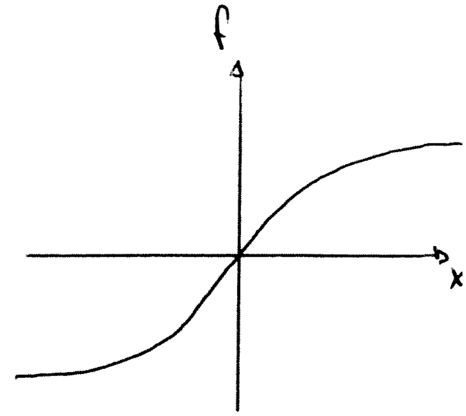
4) $f(x) = \arctan(x) = \tan^{-1}(x)$ is bounded, but still has no global maximum or minimum.

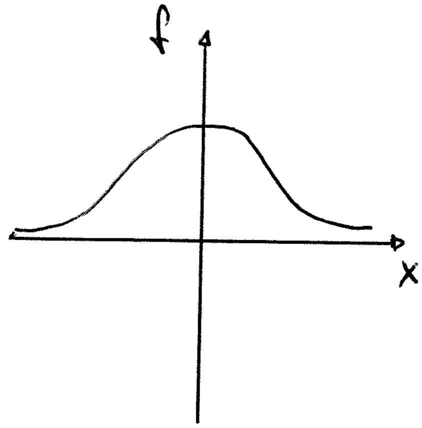
5) $f(x) = e^{-x^2}$ (Bell curve) has a global maximum of 1 at $c=0$ but no minimum.

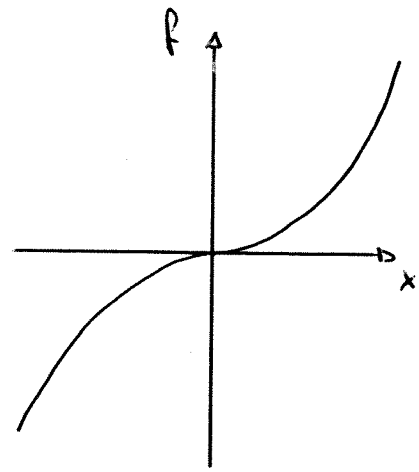
Connect the following five graphs with the examples above.











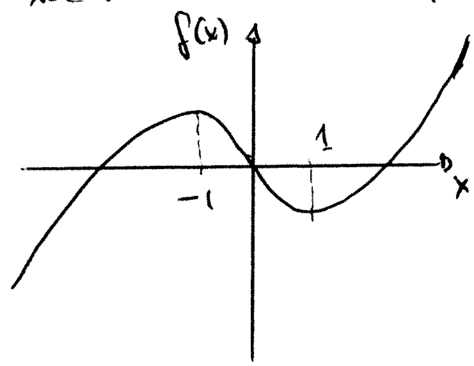
Note: Functions may or may not have extreme values. If they do, they don't have to be unique.

Definition: A relative (or local) maximum of a function f occurs at $c \in \mathrm{Dom}(f)$ if $f(x) \leq f(c)$ for all x in some interval around c.

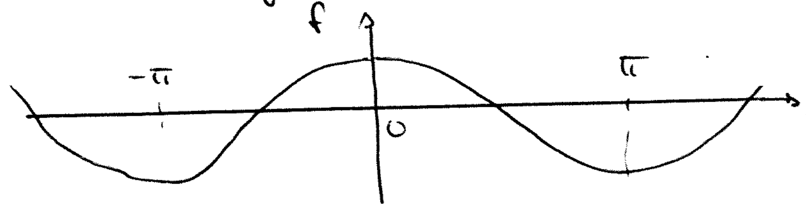
Similarly, a local minimum has $f(x) \geq f(c)$ locally.

Example: 1) $f(x) = x^3 - 3x$ (see last class)

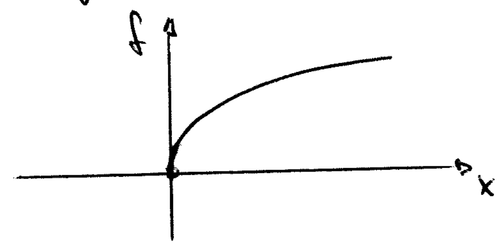
Has local min at $c=1$ and local max at $c=-1$, but no global min or max



2) $f(x) = \cos(x)$ has infinitely local max at even multiples of π and infinitely many local min at odd multiples of π.
Each local min / max is also a global min/max.



3) $f(x) = \sqrt{x}$ has no local max or min, but has a global min at $c=0$ with $f(0)=0$.
Note: Since $f(x)$ is only defined for $x \geq 0$, and since every open interval around $c=0$ contains negative numbers, we cannot check the condition for local extremum here.



Note: Local and global max/min are defined differently and behave differently. They can be the same, but they can also be very different.

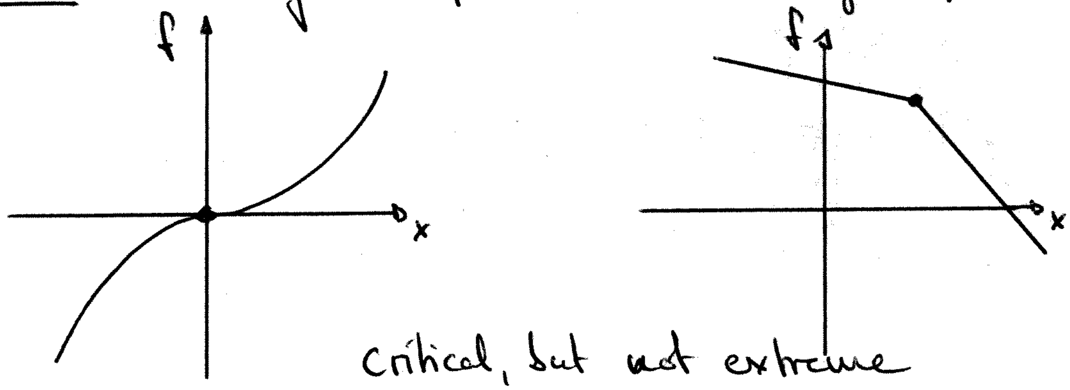
Now, how do we find these local or global extreme values?
The following theorem is of great help:
Theorem (Fermat): If f has an extreme value at c and if $f'(c)$ exists, then necessarily $f'(c) = 0$.

In other words, an extreme value can only occur at a critical point, ie $f'(c) = 0$ or $f'(c)$ does not exist.

This insight leads us to the following recipe for finding extreme values:

1) Find all critical points
2) Check each critical point (sign change of f', value of f'', ...)
3) Calculate the value of f at each critical point <u>and</u> at domain boundary.
4) Compare.

<u>Remember:</u> Not every critical point is a (local or global) extreme point



critical, but not extreme

<u>Example:</u> 1) $f(x) = \sqrt{x}\, e^{-x}$. Find local / global max/min. $\mathrm{Dom}(f) = \{x \geq 0\}$

$$f'(x) = \left(\frac{1}{2\sqrt{x}} - \sqrt{x}\right) e^{-x}$$

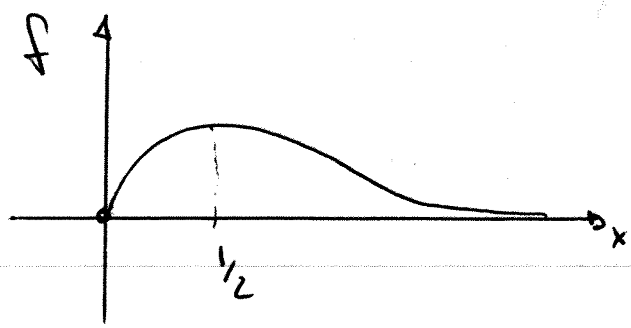
Critical point are: $x = 0$ ($f'(0)$ undefined)

$x = \frac{1}{2}$ ($f'(\frac{1}{2}) = 0$)

Since $\frac{1}{2\sqrt{x}}$ is decreasing and $\sqrt{x}$ is increasing, the sign of f' changes from + to − at $x = \frac{1}{2}$ $\Rightarrow$ local max, even global.

Since $f(0) = 0$ and $f(x) \geq 0$ for all x $\Rightarrow$ 0 is global min

$\Rightarrow$ f has global (but not local) min of 0 and global and local max at $\frac{1}{2}$.



Note: Better not calculate f'' if not necessary.

Example 2) $f(x) = |x|$ for $-1 \le x \le 2$

$f'(x)$ is not defined at $x=0$ → only critical point.

$f'(x) < 0$ for $x<0$ and $f'(x) > 0$ for $x>0$.

$f(0) = 0$ ⟹ local min.

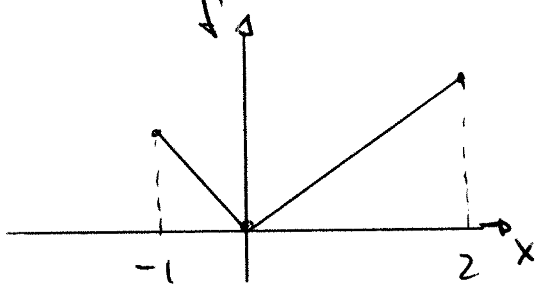
$f(x) = |x| \ge 0$ for all x ⟹ $f(0)=0$ is global min.

$f(-1) = 1$

$f(2) = 2$

Since $2>1$ ⟹ $f(2)=2$ is global max

Since -1 is at the boundary of the domain, it cannot be local max.



Example 3) $f(x) = \dfrac{\sqrt{x}}{1+x}$ $x \ge 0$ note $f(x) \ge 0$

$$f'(x) = \frac{1}{(1+x)^2}\left[\frac{1}{2\sqrt{x}}(1+x) - \sqrt{x}\right]$$

$f'(x)$ undefined at $x=0$

$f'(x) = 0$ if $1+x = 2x$ or $x=1$

Note: Calculating f'' could be a source of mistakes here.

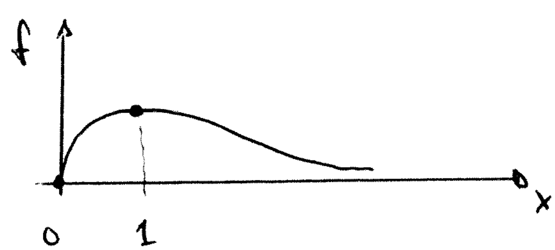
Easier to calculate: $f'(x) > 0 \iff \dfrac{1+x}{2\sqrt{x}} > \sqrt{x}$

$\iff 1+x > 2x$

$\iff x < 1$

So, $f'(x) > 0$ if $x<1$ and

$f'(x) < 0$ if $x > 1$.



Finally: can we tell a priori when extrema exist?

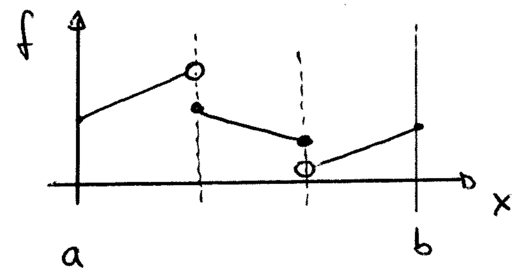
<u>Theorem</u> [extreme value theorem]

If f is continuous on the closed interval $[a,b]$, then f assumes a maximum and a minimum on $[a,b]$.

global global

Seems obvious? Consider this:

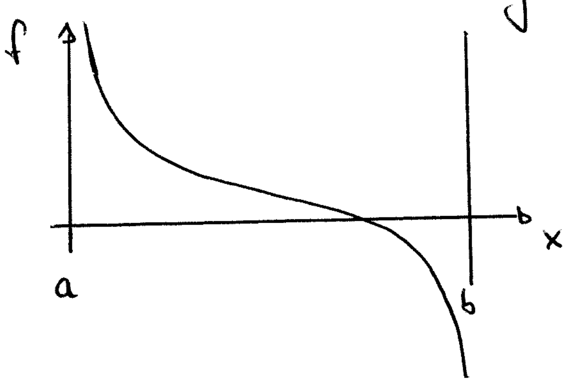
1) If f is not continuous, then the statement is wrong:



no maximum, no minimum.

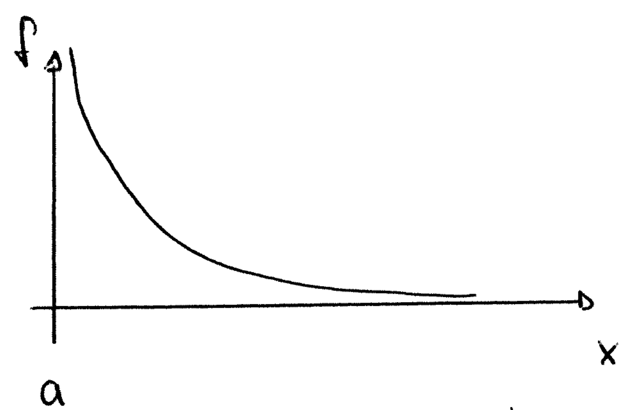
2) If f is continuous, but the interval is not closed, then the statement is wrong:

a)



no maximum, no minimum

b)



no maximum, no minimum.

here b is infinite

13 Optimization

GOAL: Apply our knowledge of extreme values to find "best" values.

Almost all optimization questions include some kind of trade-off. Sometimes, this trade-off is explict and quite obvious, sometimes it is hidden. Uncovering this trade-off is always helpful in finding and interpreting the result.

We practice several word problems here. There is no new theory. Some of these problems are quite simple, some are more abstract, and the last two are quite hard. Not sure that I will get to them in class.

1. Maximization with trade-offs: The yield of crop in agriculture changes with the amount of fertilizer (for example nitrogen) applied. When nitrogen levels in the soil are low, then adding some nitrogen will greatly increase yield. When nitrogen levels are already very high, however, adding more might decrease yield. Assume that yield as a function of nitrogen is given by the equation

$$Y(N) = \frac{N}{1+N^2}.$$

What is the opimal level of nitrogen in the soil?

$$Y'(N) = \frac{1+N^2-2N^2}{(1+N^2)^2} = \frac{1-N^2}{(1+N^2)^2} \qquad Y'(N)=0 \text{ if } N=\pm 1$$

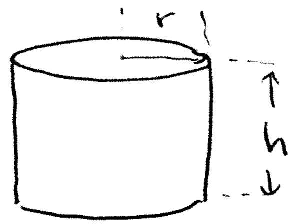
Since we are interested only in $N \geq 0$, only $N=1$ is critical.

Furthermore $Y'(N)<0$ if $N>1$ and $Y'(N)>0$ if $0<N<1$.

$\Rightarrow N=1$ is local and global maximum.

Maximum yield occurs for $N=1$ and is $Y=\frac{1}{2}$.

2. Areas and volumes: Minimize the material used to produce a cylindrical can of a fixed volume.



Approach: Denote by r the radius of the bottom of the can and by h its height. Then the volume is $V = \pi r^2 h$ and the surface area is $A = 2\pi r h + 2\pi r^2$. Use the constant volume condition to replace $h = V/(\pi r^2)$ and minimize the function

$$A(r) = \frac{2V}{r} + 2\pi r^2.$$

Find the critical value of r and the minimal surface area.

$$A'(r) = -\frac{2V}{r^2} + 4\pi r. \qquad A'(r) = 0 \iff 4\pi r = \frac{2V}{r^2}$$

$$\iff r = \sqrt[3]{\frac{V}{2\pi}}$$

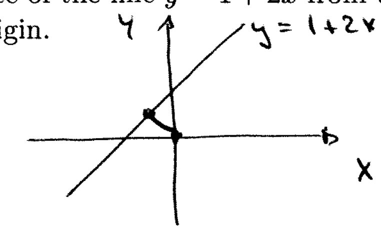
$$A''(r) = \frac{4V}{r^3} + 4\pi > 0 \text{ for all } r > 0.$$

$\Rightarrow$ A is concave up for all $r > 0$

$\Rightarrow$ $r = \sqrt[3]{\frac{V}{2\pi}}$ is local and global minimum.

$A\left(\sqrt[3]{\frac{V}{2\pi}}\right) = \ldots = (2\pi)^{1/3} V^{2/3}$ is the minimum amount of material needed to make a can of volume V.

3. Distances: Find the distance of the line $y = 1 + 2x$ from the origin and find the point on the line that is closest to the origin.



Answer: The distance of any point (x, y) from the origin is $d = \sqrt{x^2 + y^2}$. If the point is on the line, then $d = \sqrt{x^2 + (1 + 2x)^2} = \sqrt{5x^2 + 4x + 1}$. We need to minimize this function. Equivalently, we may minimize the function

$$f(x) = d^2(x) = 5x^2 + 4x + 1.$$

Since the square root is a monotone function, we will get the same location of extreme values.

$f'(x) = 10x + 4$, $f'(x) = 0$ if $x = -\frac{2}{5}$, $f''(x) = 10 > 0$

$\Rightarrow x = -\frac{2}{5}$ is local and global minimum (f is concave up everywhere).

The distance is $d(-\frac{2}{5}) = \sqrt{5 \cdot \frac{4}{25} - \frac{4 \cdot 2}{5} + 1} = \frac{1}{\sqrt{5}}$

4. Optimize food intake by adjusting residence time. Suppose that a bee remains at each flower for a fixed amount of time before it travels to the next flower. If that residence time is small, then the bee might leave valuable nectar behind. If it is large, then it might have depleted all the nectar and lost valuable time to look for the next flower. What is the optimal residence time?

Approach: To answer this question, we need to know how much food the bee collects in t time units. Let's call this function $F(t)$. If the bee takes on average τ time units to fly to the next flower, then the rate of nectar collection is

$$R(t) = \frac{F(t)}{t+\tau}.$$

Typically, F should be a positive, non-decreasing function. An example is $F(t) = t/(t+0.5)$. The bee wants to maximize R. The result is the *marginal value theorem*: the bee should leave the lower if the instantaneous food intake falls below the average food intake.

Critical points of R: $R' = \dfrac{F'(t)(t+\tau) - F(t)}{(t+\tau)^2}$

Set $R'=0$: $F'(t)(t+\tau) = F(t)$ or $F'(t) = \dfrac{F(t)}{(t+\tau)} = R(t)$

So the critical point of R is the point where $F'(t) = R(t)$,

and $R'(t) > 0 \iff F'(t) > R(t)$

$\therefore$ We don't need to differentiate R, it is enough to differentiate F.

The critical point is when the instantaneous food uptake rate (F') equals the average rate (R). The bee should leave if $F'(t) < R(t)$.

For the example $F(t) = \dfrac{t}{t+\frac{1}{2}}$, we find $F'(t) = \dfrac{1}{2(t+\frac{1}{2})^2}$

And $F'=R \iff \dfrac{1}{2(t+\frac{1}{2})^2} = \dfrac{t}{(t+\frac{1}{2})(t+\tau)}$ occurs when $t = \sqrt{\frac{\tau}{2}}$

5. Maximize yield in a DTDS. Assume that a population grows logistically and is being harvested according to the DTDS

$$x_{t+1} = 2.5x_t(1 - x_t) - hx$$

where $h > 0$ denotes the intensity of harvesting. At steady state x^* the yield is $Y(h) = hx^*$. If we harvest very little, then the yield is small. If we harvest lots, then the steady state population is small and therefore the harvest. How should we choose h so that Y is maximized?

1) Calculate x^*. $\quad x^* = 2.5x^*(1-x^*) - hx^*$

One solution is $x^*=0$. The other is given by $1 = 2.5(1-x^*)-h$

or $x^* = \dfrac{1.5-h}{2.5}$. We call this $x^*(h) = \dfrac{1.5-h}{2.5}$

2) Maximize $Y(h) = hx^*(h) = h\dfrac{1.5-h}{2.5}$

This is a parabola, it has a unique global max at $h=0.75$.

$\Rightarrow$ The optimal harvesting rate is $h = 0.75 = \frac{3}{4}$

$\Rightarrow$ The maximal harvest is $Y^* = Y(\frac{3}{4}) = \frac{3}{4}\dfrac{\frac{3}{2}-\frac{3}{4}}{\frac{5}{2}} = \dfrac{9}{40}$

6. Optimal age of reproduction. Semelparous organisms (what is this? look it up!) reproduce only once in their lifetime. Typically, they can produce more female offspring as they get older. But if they wait too long, then they might die before they reproduce. What then is the optimal age of reproduction? If we denote by $l(x)$ the probability that an individual lives to age x and by $m(x)$ the average number of female offspring of an individual at age x, then the average annual reproduction is given by

$$r(x) = \frac{\ln(l(x)m(x))}{x}. \qquad a, b, c > 0$$

Use $l(x) = e^{-ax}$ and $m(x) = bx^c$ and find the maximum of r.

1) $r(x) = \frac{1}{x}\left(\ln(e^{-ax} \cdot bx^c)\right) = -a + \frac{\ln b}{x} + c\frac{\ln x}{x}$

2) Differentiate:

$$r'(x) = -\frac{\ln b}{x^2} + c\,\frac{1-\ln x}{x^2} = \frac{1}{x^2}\left[c - \ln(bx^c)\right]$$

3) Find critical points

$$r'(x) = 0 \iff \ln(bx^c) = c \iff bx^c = e^c \iff x = e b^{-c}$$

4) Since $\ln(bx^c)$ is an increasing function of x, the derivative r' changes sign exactly once, from + to − at $x = eb^{-c}$.

Therefore, we have a global and local maximum.

7. Optimal clutch size. If an organism produces only few offspring, then each has a high probability of survival; if there are many offspring then the survival probability individually declines (look up: r-versus K strategy!). At how many offspring is the total number of survivors maximized?

Approach: Let R denote the total resources (per adult female) for reproduction and N the clutch size. Then the resource per offspring is $x = R/N$. Denote the survival probability of an offspring with resource x as $f(x)$. This function should be positive and non-decreasing. Then the expected number of surviving offspring is

$$w(x) = Nf(x) = \frac{R}{x}f(x).$$

To maximize $w(x)$, we calculate that $w'(x) = 0$ exactly if $xf'(x) = f(x)$. And if $w'(\hat{x}) = 0$ then $w''(\hat{x}) < 0$ exactly if $f''(\hat{x}) < 0$. For an example, choose $f(x) = \frac{x^2}{x^2+k^2}$.

1) $w'(x) = -\frac{R}{x^2}f(x) + \frac{R}{x}f'(x)$

2) $w'(x) = 0 \iff \frac{R}{x^2}f(x) = \frac{R}{x}f'(x) \iff f'(x) = \frac{f(x)}{x}$

3) $$w''(x) = \frac{2R}{x^3}f(x) - \frac{2R}{x^2}f'(x) + \frac{R}{x}f''(x)$$

$$= \frac{2R}{x^2}\underbrace{\left(\frac{f(x)}{x} - f'(x)\right)}_{=0 \text{ if } w'(x)=0} + \frac{R}{x}f''(x)$$

So, at the critical point, we have $w'(x)=0$ and

$\text{sign}(w''(x)) = \text{sign}\, f''(x)$

4) For $f(x) = \frac{x^2}{x^2+k^2}$ we have $w'(x) = 0 \iff x = k$

and $f''(k) = \frac{-4k^2}{(2k^2)^3} < 0$

So $x = k$ is a local max of $w(x)$.

13.1 Practice makes progress

I strongy recommend that you read through section 6.2 in the second edition of the book. It contains three detailed optimization problems and lots of detailed practice problems for those three. In the first edition, the corresponding material is scattered a little, but there is some at the end of 5.1 For general practice problens, please go to the previous section and look at the practice problems annotated 'Applications'.

Question 1: A company harvests fish at some rate $h \geq 0$. The yield is $Y(h) = h(500 - h)$ tons of fish, the selling price is \$200 per ton. The cost for harvesting at rate h is $C(h) = 1000h(1 + 0.1h)$ in dollars.
(a) Find the expression of the profit P (= revenue - cost) as a function of harvesting rate.
(b) Find the harvesting rate that maximizes profit.
(c) Find the maximum profit.

Question 2: Find the point on the curve $y = \sqrt{x}$ closest to the point (10,0). Hint: minimize the square of the distance from (10,0) to (x, y).

Question 3: In a movie theatre, the screen on the wall is 20 m high and its base is 10 m above eye level. Let θ denote the viewing angle of the screen, that is, the angle $\angle BET$ from the bottom (B) of the screen to the top (T), measured from the vertex of your eye (E). At what distance x from the screen should you position yourself to maximize θ? (from D. Kouba)

Question 4: The size of a population of bacteria introduced to a nutrient can be described by

$$N(t) = 5000 + \frac{30,000t}{100 + t^2}.$$

Find the maximum size of this population for $t \geq 0$.

Question 5: When a patient takes a drug, the concentration of this drug in the blood first increases fairly quickly and then declines again. A function that describes this behaviour is $y(t) = te^{-t/2}$, where $t \geq 0$ is the time in hours after the drug is taken.
(a) How long after drug administration does the drug concentration reach its maximum value?
(b) What is the maximum concentration?

Question 6: Find the point on the parabola $y = x^2$ that is the closest to the point $(1, 2)$ in the cartesian plane.

Question 7: Consider a population that grows according to the logistic updating function and is harvested at a linear rate $h \geq 0$. The number of individuals of the species satisfies the DTDS $x_{t+1} = x_t(4 - x_t) - hx_t$.

(a) Determine all equilibria of the DTDS.
(b) Determine conditions on h such that all equilibria are biologically relevant.
(c) Determine conditions on h such that the positive equilibrium is stable.
(d) Determine conditions on h such that the positive equilibrium is unstable.
(e) Determine the value of h that maximizes the yield and state the resulting maximum yield.

Question 8: A golf ball hit with an angle of θ radians and initial velocity of 10m/s will fly for a distance of $d(\theta) = 20.41\sin(\theta)\cos(\theta)$ metres before it lands (neglecting air resistance). Find the angle θ^* between 0 and $\pi/2$ radians that maximizes the distance flown, and find the maximal distance.

Question 9: The oxygen concentration in a lake over a single day is given by the equation

$$C(t) = 10t^3 - 120t^2 + 210t + 12000,$$

where time, $0 \leq t \leq 24$, is measured in hours. When is the oxygen concentration highest, when is it lowest. What are the maximum and minimum values?

Question 10: When a disease appears in a population, health authorities record the number of infected people. One function that describes this quantity is $y(t) = 80t^2e^{-t}$, where $t \geq 0$ is the time in days and y is the number (in units of thousands of people) of infected people.
(a) At what time will there be the most infected people and how many are there at that time?
(b) When is the number of infected people increasing and when is it increasing?
(c) Identify all points of inflection of the function $y(t)$.
(d) Find any horizontal and vertical asymptotes that may exist.
(e) Draw the graph of the number of infected people as a function of time.

L'Hôpital's rule

Recall that we spent quite some time calculating limits that, if we were to simply substitute lead to expressions as $\frac{0}{0}$, $\frac{\infty}{\infty}$, or $0\cdot\infty$, and for which we need special ideas. So far, these ideas were all <u>algebraic</u>, i.e. manipulate the expression until something simplifies and direct substitution becomes possible. Here, we apply our new tool of <u>differentiation</u> to calculate some such limits that we cannot do algebraically.

<u>Motivation 1</u>: Find $\lim_{x\to 0^+} x^2 \ln x$. We know $\lim_{x\to 0^+} x^2 = 0$, $\lim_{x\to 0^+} \ln(x) = -\infty$.

But what should $0\cdot\infty$ be? Which of the two is stronger?

<u>Motivation 2</u>: Find $\lim_{x\to\infty} \frac{ax^2-7}{x^2+1}$. We know $\lim_{x\to\infty}(ax^2-7) = \infty$

and $\lim_{x\to\infty}(x^2+1) = \infty$

But what should $\frac{\infty}{\infty}$ be?

In fact, we already know algebraically

$$\lim_{x\to\infty} \frac{ax^2-7}{x^2+1} = \lim_{x\to\infty} \frac{a - 7/x^2}{1+1/x^2} = a$$

So, $\frac{\infty}{\infty}$ could be any number, really! It depends on which is "stronger", or which of numerator/denominator goes to zero faster or slower.

Of course, we can write $x^2 \ln x = -\frac{-\ln(x)}{\frac{1}{x^2}}$ so that we have the situation of $-\frac{\infty}{\infty}$ again.

Let's formalize this:

We say that a limit is of <u>indeterminate form</u> if it can be written as

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

and one of the two following cases hold

1) $\lim_{x \to a} f(x) = \infty$ <u>and</u> $\lim_{x \to a} g(x) = \infty$

or 2) $\lim_{x \to a} f(x) = 0$ <u>and</u> $\lim_{x \to a} g(x) = 0$

Now, L'Hôpital's theorem tells us how to compute these limits.

<u>Theorem</u>: Let f, g be differentiable functions and assume that the limit $\lim_{x \to a} \frac{f(x)}{g(x)}$ is indeterminate.

Then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$ if the latter exists.

<u>Example 1</u>:

$$\lim_{x \to \infty} \frac{ax^2 - 7}{x^2 + 1} = \lim_{x \to \infty} \frac{2ax}{2x} = \lim_{x \to \infty} a = a$$

$f(x) = ax^2 - 7$, $f'(x) = 2ax$

$g(x) = x^2 + 1$, $g'(x) = 2x$

<u>Example 2</u>:

$$\lim_{x \to 0^+} x^2 \ln x \underset{\substack{\uparrow \\ \text{indeterminate form}}}{=} -\lim_{x \to 0^+} \frac{-\ln x}{\frac{1}{x^2}} \underset{\substack{\uparrow \\ \text{L'Hôpital}}}{=} -\lim_{x \to 0^+} \frac{-\frac{1}{x}}{-\frac{2}{x^3}} \underset{\substack{\uparrow \\ \text{simplify}}}{=} -\lim_{x \to 0^+} \frac{x^2}{2} \underset{\substack{\uparrow \\ \text{direct substitution}}}{=} 0$$

Example 3: $\lim_{x\to\infty} x^2 e^{-x} = \lim_{x\to\infty} \frac{x^2}{e^x} = \lim_{x\to\infty} \frac{2x}{e^x} = \lim_{x\to\infty} \frac{2}{e^x} = 0$

write in indeterminate form

$\frac{\infty}{\infty}$ use L'Hôpital

still $\frac{\infty}{\infty}$ use L'Hôpital again

$e^x \to \infty$ but $2 \to 2$ so $\frac{2}{e^x} \to 0$

→ This is true more generally: $\lim_{x\to\infty} x^n e^{-x} = 0$ for all $n = 1, 2, 3, \dots$

⇒ The exponential function e^x grows faster than <u>any</u> polynomial!

Example 4: $\lim_{x\to 0^+} \left(\frac{1}{x^2} - \frac{1}{\tan x} \right) = ?$ $\quad \infty - \infty$?

Common denominator:

$$= \lim_{x\to 0^+} \frac{\tan(x) - x^2}{x^2 \tan(x)} = \lim_{x\to 0^+} \frac{\frac{1}{\cos^2 x} - 2x}{2x\tan x + \frac{x^2}{\cos^2 x}} = \infty$$

$\frac{0}{0}$ $\qquad$ $\frac{1-0}{0+0}$

Example 5: $\lim_{x\to\infty} \frac{e^x + e^{x/2}}{e^{x/3} + e^x} = \lim_{x\to\infty} \frac{e^x + \frac{1}{2}e^{x/2}}{\frac{1}{3}e^{x/3} + e^x} = \lim_{x\to\infty} \frac{e^x + \frac{1}{4}e^{x/2}}{\frac{1}{9}e^{x/3} + e^x} =$

indeterminate $\frac{\infty}{\infty}$ use L'Hôpital

still indeterminate $\frac{\infty}{\infty}$ use L'Hôpital again

?

Note: There are cases that are in indeterminate form, but for which L'Hôpital's rule does not work! Recall the previous techniques: Divide by the fastest growing function:

$$\lim_{x\to\infty} \frac{e^x + e^{x/2}}{e^{x/3} + e^x} = \lim_{x\to\infty} \frac{e^x(1 + e^{-x/2})}{e^x(e^{-\frac{2x}{3}} + 1)} = \lim_{x\to\infty} \frac{1 + e^{-x/2}}{1 + e^{-2x/3}} = 1$$

Example 6: $\lim_{x \to 0^+} x^x = ?$

Now, this is not of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ but 0^0. Let's think about it first. For any $x > 0$: $0^x = 0$. But also for $x > 0$: $x^0 = 1$. Now what should 0^0 be?

Write this expression in a form so that L'Hôpital applies. $x^x = e^{x \ln x}$

and $x \ln x = -\frac{-\ln x}{\frac{1}{x}}$ $\longrightarrow$ this is of indeterminate form $-\frac{\infty}{\infty}$ as $x \to 0^+$

Now:

$$\lim_{x\to 0^+} x^x \underset{\text{use } x = e^{\ln x}}{=} \lim_{x \to 0^+} e^{x \ln x} \underset{\text{continuity}}{=} \exp\left(-\lim_{x\to 0^+} \frac{-\ln x}{\frac{1}{x}}\right) \underset{\substack{\text{indeterminate } \frac{\infty}{\infty} \\ \text{use L'Hôpital}}}{=} \exp\left(-\lim_{x \to 0^+} \frac{-\frac{1}{x}}{-\frac{1}{x^2}}\right) =$$

$$= \exp\left(-\lim_{x\to 0^+} x\right) = e^{-0} = 1.$$

Now to practice at home: use this information to graph the function $f(x) = x^x$ for $x > 0$.

<u>Example 7:</u> $\lim_{x\to\infty}\left(1+\frac{y}{x}\right)^x = ?$ $\quad y\in\mathbb{R}$ is fixed. Any number

Again, this is not in the standard form, so let's see. As $x\to\infty$ we have $\left(1+\frac{y}{x}\right)\to 1$ and we know that $1^n = 1$ for all n.

But if a is any number >1, then $a^x \to \infty$ as $x\to\infty$. So, the limit could be anything between 1 and ∞ if $y>0$. Let's see.

First step: $\left(1+\frac{y}{x}\right)^x = \exp\left(x\cdot\ln\left(1+\frac{y}{x}\right)\right)$ as before in Example 6.

Second step: Let's look only at the exponent: $x\cdot\ln\left(1+\frac{y}{x}\right)$

As $x\to\infty$, we have $\ln\left(1+\frac{y}{x}\right)\to\ln(1)=0$.

So, the product is of the form $0\cdot\infty$, which we can get into indeterminate form easily: $x\cdot\ln\left(1+\frac{y}{x}\right) = \dfrac{\ln\left(1+\frac{y}{x}\right)}{\frac{1}{x}} \to \frac{0}{0}$

Third step: Take the limit

$$\lim_{x\to\infty} x\ln\left(1+\frac{y}{x}\right) = \lim_{x\to\infty}\frac{\ln\left(1+\frac{y}{x}\right)}{\frac{1}{x}} \underset{\text{L'Hôpital}}{=} \lim_{x\to\infty}\frac{\frac{1}{1+y/x}\cdot\left(-\frac{y}{x^2}\right)}{-\frac{1}{x^2}} =$$

$$= \lim_{x\to\infty}\frac{y}{1+y/x} = y$$

Finally: Put it all together

$$\lim_{x\to\infty}\left(1+\frac{y}{x}\right)^x = \lim_{x\to\infty}\exp\left(x\ln\left(1+\frac{y}{x}\right)\right) = \exp\left(\lim_{x\to\infty}\frac{\ln\left(1+\frac{y}{x}\right)}{\frac{1}{x}}\right) = e^y$$

So, here is a way to define the exponential function as a limit.

Question: Which number is larger: e^{π} or π^{e}?

Find a solution that does not require a calculator.

Think first: $e \simeq 2.78$, $\pi \simeq 3.14$: similar in value.

So, the two expressions could be similar.(*)

Solution: Define the function $f(x) = e^{-x} x^{e}$. Then $f(\pi) = e^{-\pi}\pi^{e}$

So, if $f(\pi) > 1$ then $e^{-\pi}\pi^{e} > 1$ or $\pi^{e} > e^{\pi}$

But if $f(\pi) < 1$ then $e^{-\pi}\pi^{e} < 1$ or $\pi^{e} < e^{\pi}$.

Let's check f. $f(0) = 0$, $\lim_{x \to \infty} f(x) = 0$ (see above).

$$f'(x) = -e^{-x}x^{e} + e x^{e-1} e^{-x} = e^{-x} x^{e-1}(e - x)$$

So $f'(x) > 0$ if $x < e$ and $f'(x) < 0$ if $x > e$.

f has a local and global maximum at $x = e$

$$f(e) = e^{-e} e^{e} = 1.$$

So $f(\pi) < f(e) = 1$ and so $\pi^{e} < e^{\pi}$

(*) In fact, lets look at two examples:

$2 < 3$	$3 < 4$
$2^3 = 8 < 3^2 = 9$	$3^4 = 81 > 4^3 = 64$

Polynomial approximation

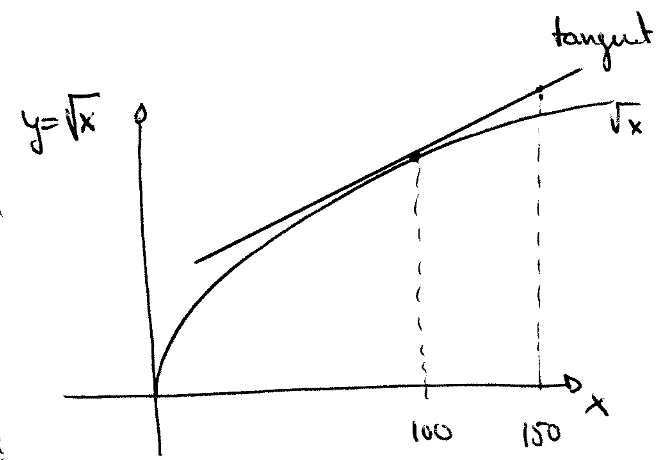
<u>Motivation</u>: Many functions are quite difficult to evaluate (e.g. $\sqrt{x}$, $\ln(x)$, ...), but some functions are easy (e.g. linear, quadratic). Can we use these simpler functions to approximate the complicated ones?

For example, if I want to know the value of $\sqrt{150}$. I can think of $\sqrt{100}=10$, $\sqrt{144}=12$, $\sqrt{169}=13$. I also know that the graph of $\sqrt{x}$ is monotone. So $\sqrt{150}$ should be somewhere between 12 and 13. But where

<u>Goal</u>: Find ways to systematically approximate complicated functions.

<u>First idea</u>: Tangent line.

Because I know the value of $\sqrt{x}$ for some x, I can easily write down a tangent line in these points. And I can use the tangent line as an approximation to the function.



<u>Formally</u>: If $y=f(x)$ and some base point a are given, then the tangent line to f at a is $T(x) = f(a) + f'(a)(x-a)$

<u>Example</u>: $f(x)=\sqrt{x}$, $a=100$, then $f(a)=\sqrt{100}=10$, $f'(a)=\frac{1}{2\sqrt{a}}=\frac{1}{20}$

and so $T(x) = f(a)+f'(a)(x-a) = 10+\frac{1}{20}(x-100)$

we can evaluate $T(150) = 10+\frac{50}{20} = 12.5$. [Check: $\sqrt{150}=12.247$] not bad!

We expect that if we choose the base point closer to 150, then the approximation could be even better. Let's choose $a = 144$.

Then $f(a) = \sqrt{144} = 12$, $f'(a) = \dfrac{1}{2\sqrt{144}} = \dfrac{1}{24}$

$T(x) = 12 + \dfrac{1}{24}(x - 144)$ and $T(150) = 12 + \dfrac{6}{24} = 12 + \dfrac{1}{4} = 12.25$.

Really good!

Try base point $a = 169$ at home!

<u>Observation</u>: In both cases, the approximate value from the tangent line is greater than the true value. Looking at the graph of $y = \sqrt{x}$, this is clear: Since the graph is concave down, the tangent line is above the true value. If the graph were concave up, the tangent line approximation would underestimate the true value.

<u>Example</u>: What is $e^{1/2}$? Look at $f(x) = e^x$, $f'(x) = e^x$ and $f(0) = 1$

Then with base point $a = 0$, we get the tangent line

$T(x) = f(0) + f'(0)(x - 0) = 1 + x$ so that $T(\frac{1}{2}) = \frac{3}{2} = 1.5$

by calculator: $e^{1/2} = 1.648 > 1.5$.

<u>Motivation and question</u>:

So, to make our approximation better, could we include information about the curvature of our function and not only the slope?

In other words, can we find a quadratic function that, at a given point a, has the same value as f, the same slope <u>and</u> the same curvature?

<u>Answer</u>: Yes, this is possible. In fact, we can find a polynomial of degree n, called $T_n(x)$ with the property that

$T_n(a) = f(a)$, $T_n'(a) = f'(a)$, $T_n''(a) = f''(a)$, ... $T_n^{(n)}(a) = f^{(n)}(a)$,

where $f^{(n)}$ is the n-th derivative of f.

<u>Definition</u>: The <u>Taylor polynomial</u> of a function f of degree n and base point a is given by

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{6}(x-a)^3 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

<u>Note</u>: • $f^{(n)}(a) = \frac{d^n}{dx^n} f(x)$ at $x=a$, the n-th derivative.

• $n! = 1 \cdot 2 \cdot 3 \cdot \dots \cdot n$ (factorial), so $3! = 1\cdot2\cdot3 = 6$
$4! = 1\cdot2\cdot3\cdot4 = 24$

• $T_1(x) = T(x)$ the tangent line.

• The difference between $T_n(x)$ and $T_{n-1}(x)$ is only one term, the last.

<u>Back to Example of $\sqrt{150}$</u>. Let's try a quadratic approximation, i.e. $T_2(x)$.

$$T_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$$

with $f(x) = \sqrt{x}$
$a = 100$

$$= 10 + \frac{1}{20}(x-100) - \frac{1}{2\cdot4\cdot1000}(x-100)^2$$

$f'(x) = \frac{1}{2\sqrt{x}}$

$$= 10 + \frac{x-100}{20} - \frac{(x-100)^2}{8000}$$

$f''(x) = \frac{1}{4\sqrt{x^3}}$

Note: as we had expected, the new term is negative (curvature), so that when we evaluate $T_2(150)$ it will be smaller than $T_1(150)$.

$$T_2(150) = 10 + \frac{50}{20} - \frac{50^2}{8000} = 10 + \frac{5}{2} - \frac{5}{16} = 12.1875$$

a little to small, but quite good

For base point $a = 144$, we get

$$T_2(x) = 12 + \frac{1}{24}(x-144) - \frac{1}{8\cdot12^3}(x-144)^2. \quad T_2(150) = 12.25 - \frac{36}{8\cdot12^3} = 12.25 - \frac{1}{384}$$

$= 12.247$ right

Note: • Using the quadratic approximation is much closer than the linear (tangent line) approximation.

• Being closer to the actual value is much better, but can be more difficult to calculate.

Let's see how much better we can do with a cubic approximation.

Recall: $T_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{6}(x-a)^3$

$f(x) = \sqrt{x}$

$f'(x) = \frac{1}{2\sqrt{x}}$

$f''(x) = \frac{-1}{4\sqrt{x^3}}$

$f'''(x) = \frac{3}{8\sqrt{x^5}}$

$a = 100$

$$= 10 + \frac{x-100}{20} - \frac{(x-100)^2}{8000} + \frac{3(x-100)^3}{48 \cdot 10^5}$$

$$= 10 + \frac{x-100}{20} - \frac{(x-100)^2}{8000} + \frac{(x-100)^3}{16 \cdot 10^5}$$

$$T_3(150) = 10 + \frac{1}{2} - \frac{5}{16} + \frac{5}{64} = 12.265$$ very close.

New example: Find an approximation of $(8.1)^3$, using a cubic polynomial.

Note: $8^3 = 512$ this is easy. Now choose $f(x) = x^3$ and use base point $a = 8$. Then

$f'(x) = 3x^2$, $f''(x) = 6x$, $f'''(x) = 6$. And so

$$T_3(x) = f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2} + f'''(a)\frac{(x-a)^3}{6}$$

$$= 512 + 192(x-8) + 24(x-8)^2 + \frac{6}{6}(x-8)^3$$

$$T_3(8.1) = 512 + 19.2 + 0.24 + 0.001 = 531.441$$

This is actually the exact value, i.e. $(8.1)^3 = 531.441$.

And if you think about it, $f^{(4)} = 0$, $f^{(n)} = 0$ for $n \geq 4$. So if we went to higher order T_n, we would not change the approximation.

More examples:

1) Find $T_4(x)$ for $f(x) = \cos(x)$ with base point $a=0$.

$f(0) = \cos(0) = 1$, $f'(x) = -\sin(x)$ $f'(0) = 0$, $f''(x) = -\cos x$ $f''(0) = -1$

$f'''(x) = \sin(x)$ $f'''(0) = 0$, $f^{(4)}(x) = \cos x$ $f^{(4)}(0) = 1$

$$T_4(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{6}(x-a)^3 + \frac{f^{(4)}(a)}{24}(x-a)^4$$

$$= 1 + 0(x-0) - \frac{1}{2}(x-0)^2 + 0(x-0)^3 + \frac{1}{24}(x-0)^4$$

$$= 1 - \frac{x^2}{2} + \frac{x^4}{24}$$

2) Use $T_3(x)$ to approximate the value of $\ln(1.1) \approx 0.0953$

$f(x) = \ln(x)$ with $a=1$ so that $f(1) = \ln(1) = 0$.

$f'(x) = \frac{1}{x}$ $f'(a) = 1$; $f''(x) = -\frac{1}{x^2}$ $f''(a) = -1$

$f'''(x) = +\frac{2}{x^3}$ $f'''(a) = +2$.

$$T_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{6}(x-a)^3$$

$$= 0 + 1\cdot(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$$

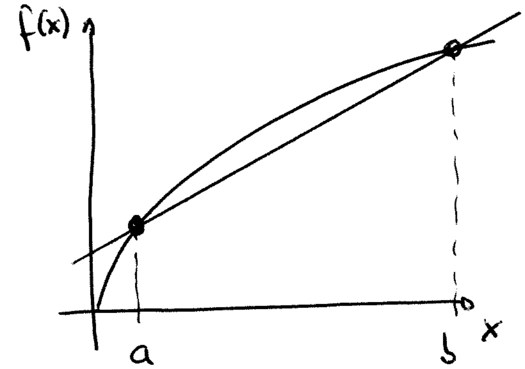
$$T_3(1.1) = 0.1 - 0.05 + \frac{0.001}{3} = 0.0953$$

3) Find a good approximation for $\sqrt{6}$. (Use $T_3(x)$ with $a=4$ and $a=9$. compare)

4) Find a good approximation for $\sin(1)$. (Use $T_3(x)$ with $a=0$)

So, if we use the information about $f, f', f'', \ldots$ at a single point, we can get a good approximation to f near that point. Away from that point, the approximation can get very bad.

Another idea is to use two points and approximate the function between the two points by a straight line. The secant line.



The slope of the secant line is

$$m = \frac{f(b) - f(a)}{b - a}$$

and the equation is

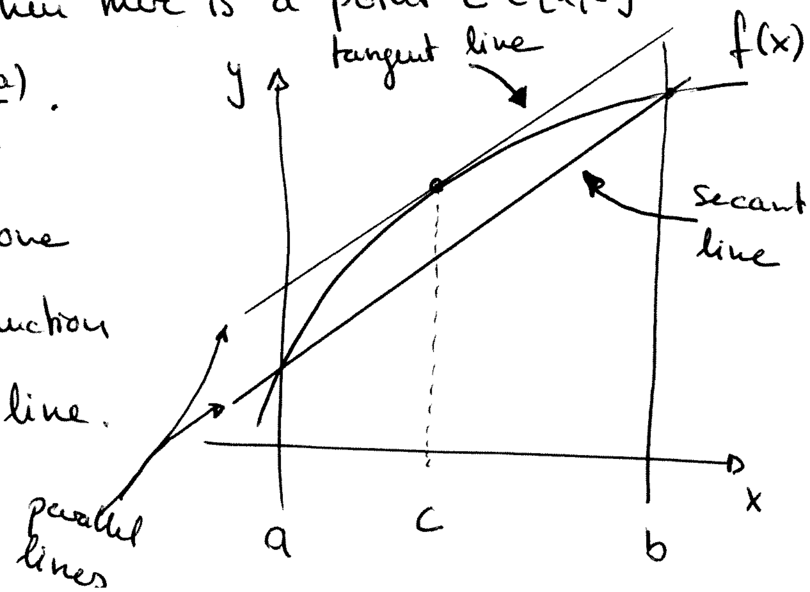
$$y = f(a) + \frac{f(b) - f(a)}{b - a}(x - a)$$

This approximation is not as good near the points a, b as the tangent line, but it is reasonably good on the entire interval $[a, b]$, if that interval is small enough.

An interesting relation between secant lines and slopes of functions is the following Mean Value theorem:

If f is differentiable on $[a, b]$, then there is a point $c \in [a, b]$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

In other words, there is at least one point, where the slope of the function equals the slope of the tangent line.



1) <u>Example</u>: $f(x) = x^3$ on $[0,1]$

The secant line between $(0, f(0))$ and $(1, f(1))$ is the line $y = x$ and has slope $1 = \frac{f(1)-f(0)}{1-0} = \frac{1-0}{1-0} = 1$.

The derivative is $f'(x) = 3x^2$, and $3x^2 = 1 \Leftrightarrow x = \pm\sqrt{\frac{1}{3}}$.

So setting $c = \sqrt{\frac{1}{3}}$ we have $f'(c) = 1$ and $a \leq c \leq b$.

2) <u>Example</u>: A car travels from Ottawa to Montréal (200 km) in 2 hours. Denote by $x(t)$ its distance from Ottawa, x in km, t in hours. Then $x(0) = 0$, $x(2) = 200$. The secant line has slope $\frac{x(2)-x(0)}{2-0} = 100$.

The derivative is $x'(t) = v(t) =$ speed of the car.

By the Mean Value Theorem, there is at least one value $t^* \in [0,2]$ so that $x'(t^*) = 100$. Therefore, the car had speed $100\,\text{km}/\text{h}$ at least once.

3) <u>Special case</u>: If $f(b) = f(a)$, then we have <u>Rolle's Theorem</u>

If f is differentiable on $[a,b]$ and $f(b) = f(a)$, then there is some $c \in [a,b]$ with $f'(c) = 0$. I.e. f has at least one critical point.

This can be really important when we cannot explicitly solve $f'(x) = 0$.

For example: $f(x) = \sin(x)\ln(x)$

$$f(\pi) = f(2\pi) = 0$$

$\Rightarrow$ There is a point $c \in [\pi, 2\pi]$ with $f'(c) = 0$, i.e. there is a critical point.

$f'(x) = \cos(x)\ln x + \frac{\sin(x)}{x}$ Find $f'(c) = 0$ explicitly? Impossible.

4) Example: estimates.

Suppose, you know $f(a)$ and $f'(x)$ and want to know something about $f(b)$. Let's start with the Mean Value Theorem again:

$\frac{f(b)-f(a)}{b-a} = f'(c)$ and solve for $f(b) = f(a) + f'(c)(b-a)$

For example, if we know that $|f'(c)| < M$ then $f(b) < f(a) + M(b-a)$
and $f(b) > f(a) - M(b-a)$

Let's go to the example of $\sqrt{150}$ again. Let's choose $a=100$ and $b=150$.
Then $b-a = 50$. $f'(x) = \frac{1}{2\sqrt{x}}$ so $f'(x)$ is maximal if x is minimal.
So $f'(x) \le \frac{1}{20}$ for $x \in [100, 150]$. Then

$$f(150) = f(a) + f'(c)(b-a) \le f(100) + \max\{f'(c)\}(150-100)$$
$$= 10 + \frac{1}{20}\cdot 50 = 12.5$$

So, we know that $\sqrt{150} \le 12.5$.

Stability of fixed points in nonlinear DTDS

Recall: A DTDS is an iteration $x_{t+1} = f(x_t)$. A fixed point satisfies $x^* = f(x^*)$.

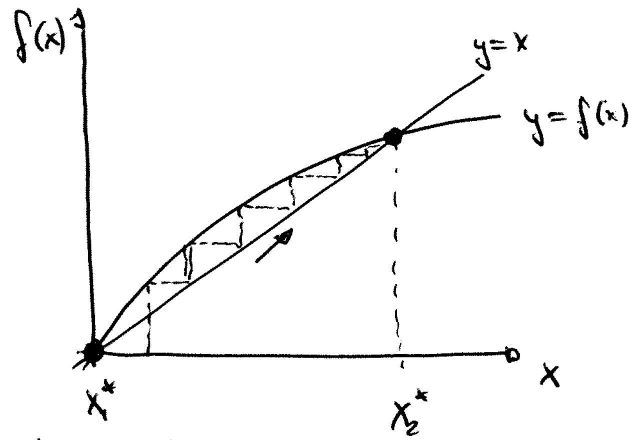
A fixed point is called stable if all nearby solutions converge to x^*, and unstable otherwise.

Method: Cobwebbing.

Example: $$x_{t+1} = f(x_t) = \frac{2x_t}{1+x_t}$$

$x_1^* = 0$ is unstable

$x_2^* = 1$ is stable

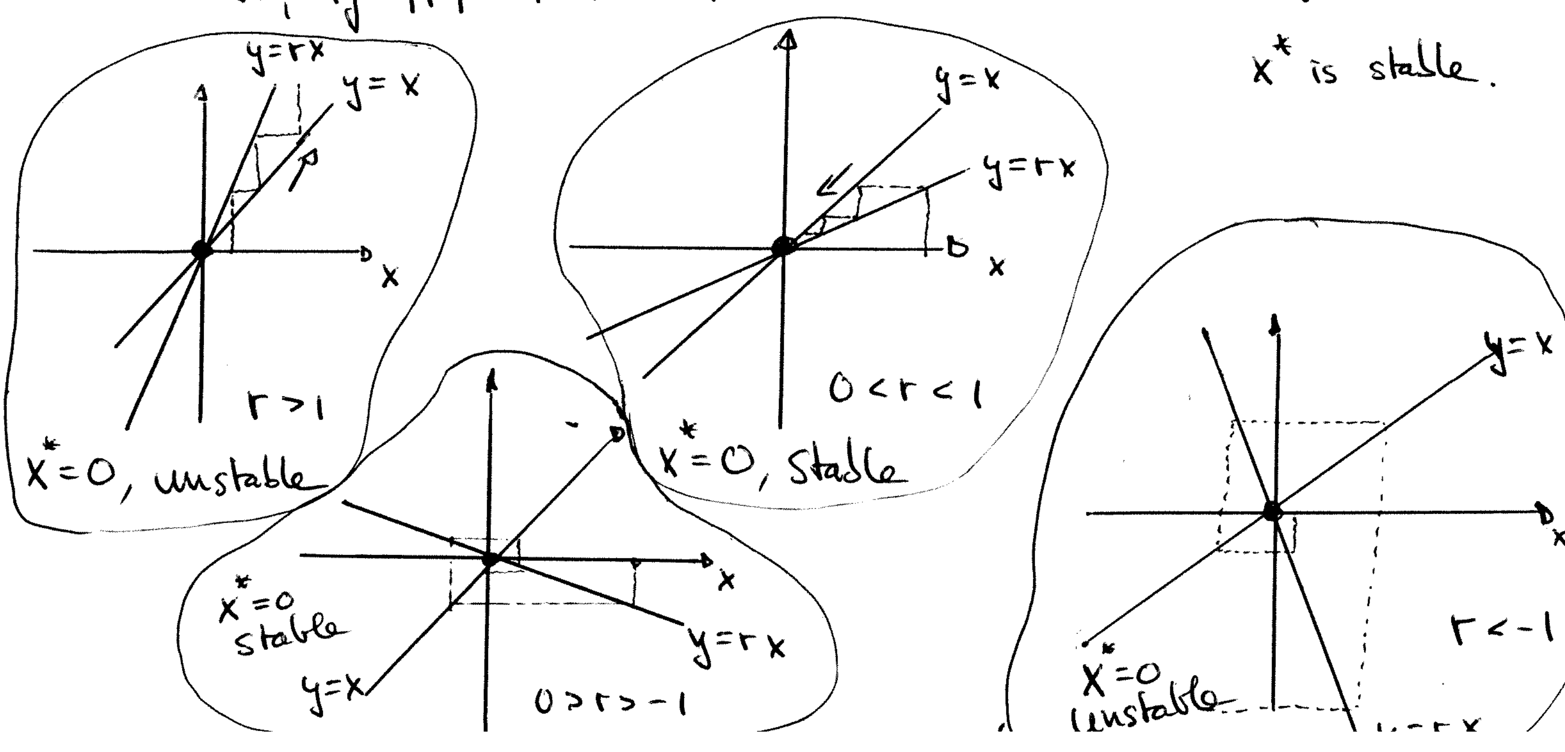


Goal: Find an analytical condition for x^* to be stable, so that we don't have to rely on graphing and cobwebbing.

Recall: A linear DTDS: $x_{t+1} = rx_t + d$ has $x^* = \frac{d}{1-r}$, $(r \neq 1)$.

It has the general solution $$x_t = r^t(x_0 - x^*) + x^*.$$

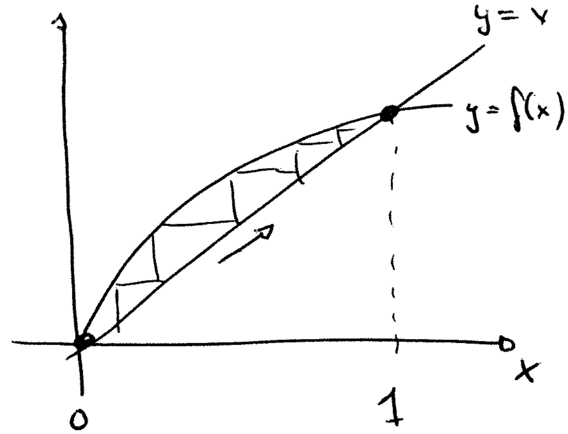
So, if $|r| < 1$ then $r^t \to 0$ as $t \to \infty$ and $x_t \to x^*$, so x^* is stable.



<u>Idea</u> for nonlinear DTDS: Replace $f(x)$ near x^* by the tangent line approximation and use it to find stability.

<u>Guess</u>: x^* is stable if $|f'(x^*)| < 1$

<u>Back to example</u>: $\quad x_{t+1} = \dfrac{2x_t}{1+x_t}$



$$f(x) = \frac{2x}{1+x}$$

$$f'(x) = \frac{2}{(1+x)^2} \qquad f'(0) = 2 > 1$$

$$|f'(1)| = \frac{1}{2} < 1$$

→ Seems to work.

<u>Theory</u>: $x_{t+1} = f(x_t)$ and $x^* = f(x^*)$. Write $x_t = x^* + y_t$, where y_t is the difference between x_t and x^*. If $y_t \to 0$ then $x_t \to x^*$ and so x^* is stable.

Now

tangent line approximation, base point x^*

$$y_{t+1} + x^* = x_{t+1} = f(x_t) \approx f(x^*) + f'(x^*)(x_t - x^*)$$

$$y_{t+1} + x^* \approx f(x^*) + f'(x^*)\, y_t$$

Since $x^* = f(x^*)$: $\quad \to \quad y_t \approx f'(x^*)\, y_t$

This is a linear equation of the form $y_{t+1} = r\, y_t$ with $r = f'(x^*)$.

So $\begin{cases} y_t \to 0 & \text{if } |f'(x^*)| < 1 \text{ and } x^* \text{ is stable} \\ |y_t| \to \infty & \text{if } |f'(x^*)| > 1 \text{ and } x^* \text{ is unstable.} \end{cases}$

Example
Allee Effect

$$x_{t+1} = f(x_t) = \frac{3x_t^2}{1+x_t^2}$$

Fixed Points: $x = f(x) = \dfrac{3x^2}{1+x^2} \Rightarrow x(1+x^2) = 3x^2 \Rightarrow x = 0$ or

$$(1+x^2) = 3x \Rightarrow x^2 - 3x + 1 = 0.$$

$$x = \frac{3}{2} \pm \sqrt{\frac{9}{4} - 1} = \frac{3}{2} \pm \sqrt{\frac{5}{4}} = \frac{1}{2}\left(3 \pm \sqrt{5}\right) \approx \begin{cases} 2.6 \\ 0.4 \end{cases}$$

$\Rightarrow$ There are three positive fixed points: $x_1 = 0$, $x_2 = \frac{1}{2}(3-\sqrt{5})$, $x_3 = \frac{1}{2}(3+\sqrt{5})$

Stability: $f'(x) = \dfrac{6x(1+x^2) - 3x^2(2x)}{(1+x^2)^2} = \dfrac{6x}{(1+x^2)^2}$

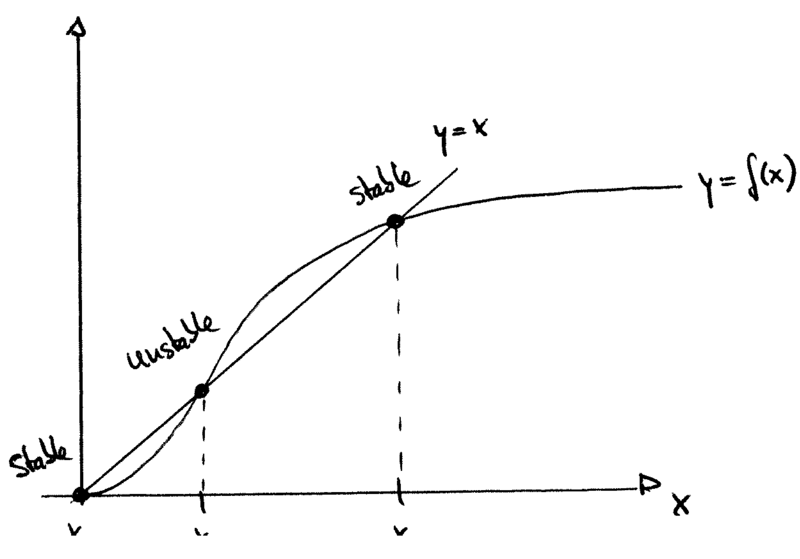
$f'(x_1) = f'(0) = 0$

At the fixed point: $1+x^2 = 3x$ (see above). Therefore

$$f'(x_2) = \frac{6x_2}{(1+x_2^2)^2} = \frac{6x_2}{(3x_2)^2} = \frac{6x_2}{9x_2^2} = \frac{2}{3x_2} = \frac{4}{3(3-\sqrt{5})} > 1$$

Similarly
$$f'(x_3) = \frac{6x_3}{(1+x_3^2)^2} = \frac{6x_3}{9x_3^2} = \frac{2}{3x_3} = \frac{4}{3(3+\sqrt{5})} < 1$$

$\Rightarrow$ $x_0 = 0$ is stable, $x_1 = \frac{1}{2}(3-\sqrt{5})$ is unstable, $x_2 = \frac{1}{2}(3+\sqrt{5})$ is stable



Example: The Ricker equation.

$$x_{t+1} = x_t e^{r(1-x_t)} = f(x_t), \quad r>0$$

Fixed points: $x = x e^{r(1-x)} \Leftrightarrow x=0$ or $1 = e^{r(1-x)}$

$\Leftrightarrow x_1^* = 0$ or $x_2^* = 1$.

There are two fixed points.

Stability: $f'(x) = e^{r(1-x)} - rxe^{r(1-x)} = (1-rx)e^{r(1-x)}$

$f'(0) = e^r > 1$ if $r>0$

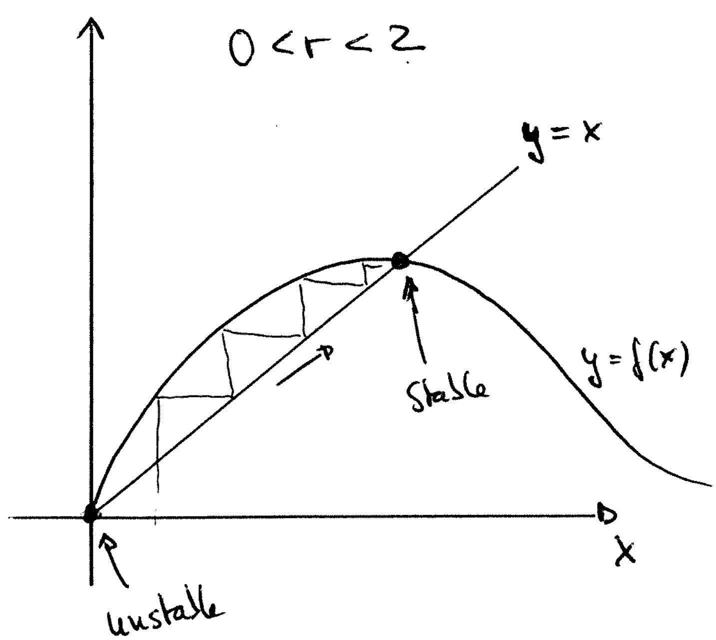
$f'(1) = 1-r$. So $|f'(1)| < 1$ if $|1-r|<1$

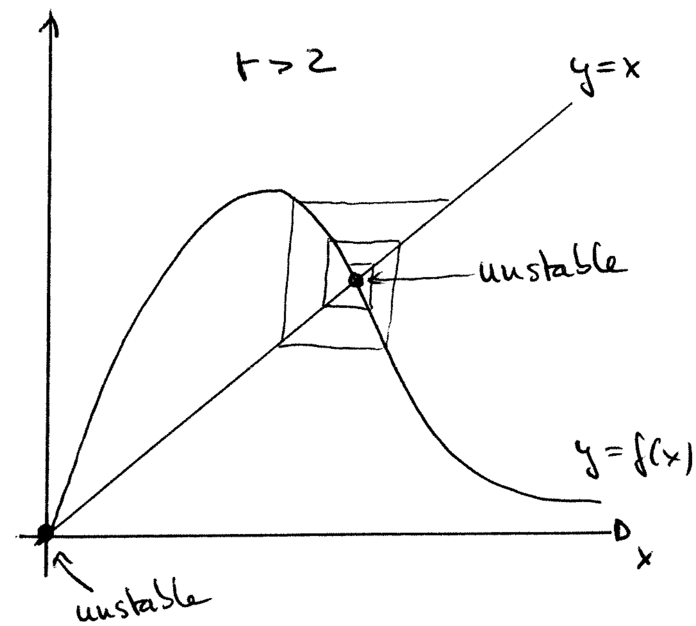
or $-1 < 1-r < 1$

or $0<r<2$

Result: $x_1^*=0$ is unstable since $r>0$.

$x_2^* = 1$ is stable if $0<r<2$ and unstable if $r>2$.





What happens if there is no stable state? Where do solutions go?

Example: Harvesting and optimization in DTDS

Consider a population growing according to $x_{t+1} = 2x_t(2-x_t)$ and then being harvested at some rate h, so that

$$x_{t+1} = f(x_t) = 2x_t(2-x_t) - hx_t, \qquad h>0$$

What are the feasible harvest rates at equilibrium?
What is the maximum yield?
Is harvesting for maximum yield stable?

1) Equilibrium: $x_1^* = 0$ or $1 = 2(2-x) - h$ so $x_2^* = \frac{3-h}{2}$

Feasible harvest rates if $x_2^* > 0$ so $0 < h < 3$

2) Maximum yield: The yield is hx^* at equilibrium or $Y = h\frac{3-h}{2}$.

To find a maximum, differentiate $Y'(h) = \frac{3}{2} - h$

critical point: $h = \frac{3}{2}$, maximum (look at shape of Y: quadratic)

$$Y\left(\tfrac{3}{2}\right) = \frac{3}{2}\cdot\frac{3}{4} = \frac{9}{8} \approx 1.125$$

3) Stability of the point $x_2^* = \frac{3-h}{2}$ of the DTDS

$$f'(x) = 4 - 4x - h$$

$$f'\left(\tfrac{3-h}{2}\right) = 4 - (6-2h) - h = h-2$$

$$\left|f'\left(\tfrac{3-h}{2}\right)\right| = |h-2| < 1 \quad \text{if } 1 < h < 3 \text{ so}$$

Since max yield is at $h = \frac{3}{2}$, the x_2^* is stable.

Summary: $x_{t+1} = f(x_t) = 2x_t(2-x_t) - hx_t \quad , h \geq 0$

Parameter	fixed point	stability
$0 \leq h < 1$	$x_1^* = 0$ $x_2^* = \frac{3-h}{2}$	unstable unstable
$1 < h < 3$	$x_1^* = 0$ $x_2^* = \frac{3-h}{2}$	unstable stable
$h > 3$	$x_1^* = 0$ $x_2^* = \frac{3-h}{2} < 0$	stable if $3 < h < 5$ not biologically relevant

---

Example: The logistic equation $x_{t+1} = f(x_t) = rx_t(1-x_t)$, $0 < r < 4$.

The equilibria are $x_1^* = 0$ and $x_2^* = \frac{r-1}{r}$. $x_2^* > 0$ only if $r > 1$.

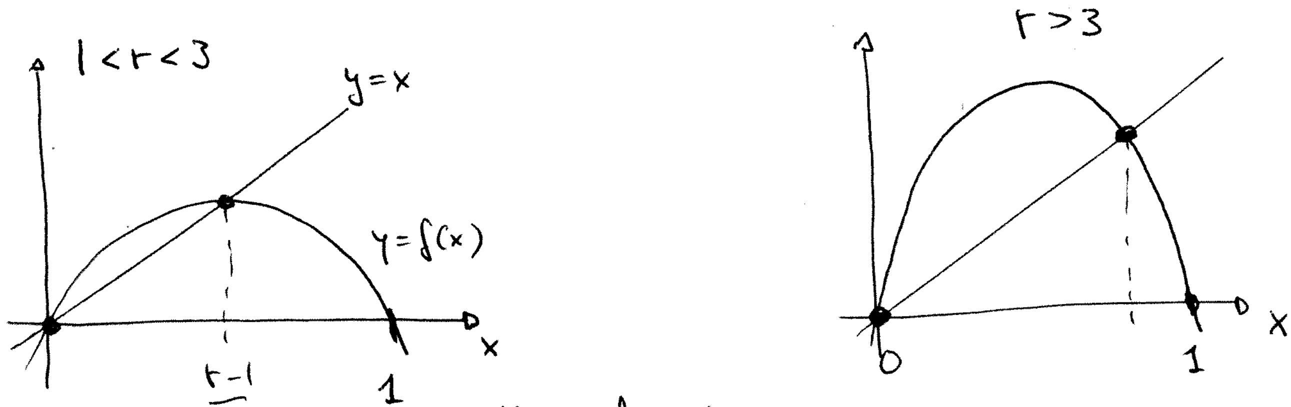
Derivative: $f'(x) = r(1-2x)$

$f'(0) = r \Rightarrow x_1^*$ is stable if $0 \leq r < 1$ and unstable if $r > 1$.

$f'(x_2^*) = f'\left(\frac{r-1}{r}\right) = 2 - r$

$|2-r| < 1$ if $-1 < 2-r < 1 \Leftrightarrow 1 < r < 3$

So x_2^* is stable if $1 < r < 3$.



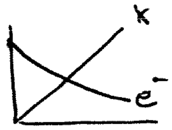
Use logisticDTDS.xls to explore.

Root finding, intermediate value theorem, and Newton's method

Motivation: Find the equilibrium of the DTDS $x_{t+1} = e^{-x_t}$.

Solve $x = e^{-x}$. No explicit solution, i.e. $x = \ldots$ (nothing with x here)

Picture:



for $x=0$, $x < e^{-x}$, for $x=1$: $x > e^{-x}$

Somewhere in between we should have $x = e^{-x}$

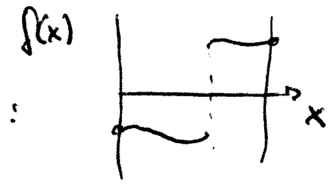
Idea: Form the function $f(x) = x - e^{-x}$. Then $f(0) = -\frac{1}{e} < 0$

$f(1) = 1 - \frac{1}{e} > 0$

Then there should be a number in between 0 and 1 with $f(x) = 0$.

Caveat:

What if f:

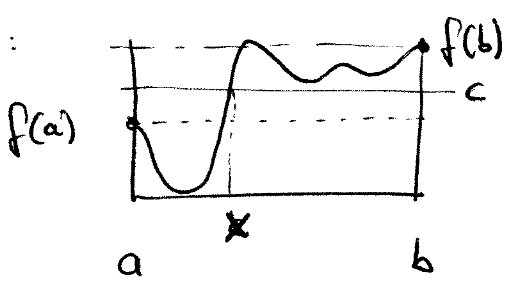


We probably need continuity.

Intermediate value theorem (IVT)

If f is continuous on $a \le x \le b$ and c is between $f(a)$ and $f(b)$, then there is a number $x \in [a,b]$ with $f(x) = c$

Illustration:



note: there could be more than one value of x with $f(x) = c$.

How does this IVT help us find the zero of a function or the equilibrium of a DTDS?

$\to$ Apply IVT repeatedly

<u>Bisection method</u>: Find a zero of $f(x) = x - e^{-x}$

First guess as above: $f(0) < 0$, $f(1) > 0$, f is continuous.

Then by IVT: There is some $x \in [0,1]$ with $f(x) = 0$.

Now evaluate midpoint: $f(\frac{1}{2}) = \frac{1}{2} - e^{-1/2} = -0.106 < 0$

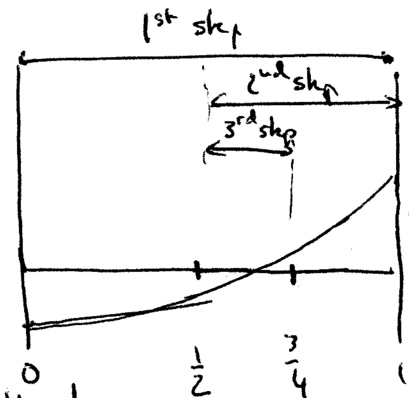
Then by IVT: There is some $x \in [\frac{1}{2}, 1]$ with $f(x) = 0$

Take the midpoint again: $f(\frac{3}{4}) = \frac{3}{4} - e^{-3/4} = 0.277 > 0$

Then by IVT: There is some $x \in [\frac{1}{2}, \frac{3}{4}]$ with $f(x) = 0$

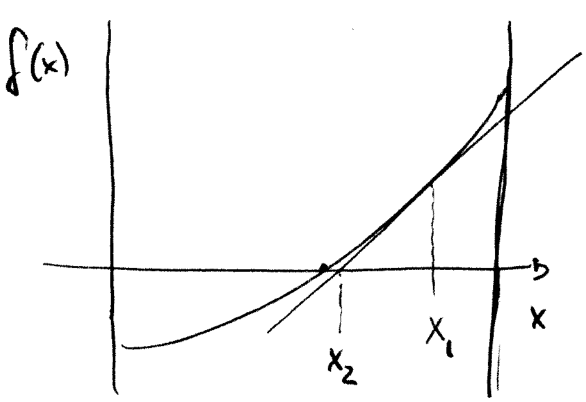
Continue until the interval is so small that you have the zero as close as you wish.

This method is ~~good~~, but slow:



<u>Faster and more elaborate: Newton's method.</u>

Uses information about derivative; uses tangent line approximation.



1) Get initial guess for x_1 so that $f(x_1) \approx 0$
2) Use tangent line approximation at that x_1
3) Find zero of tangent line, call it x_2
4) Take that zero as the next guess for $f(x) = 0$
5) Repeat.

So given $f(x)$, find tangent line at base point x_1 where $f(x_1) \approx 0$.

$$T_1(x) = y(x) = f(x_1) + f'(x_1)(x - x_1)$$

Set the tangent line to zero: $y(x) = 0$: $f(x_1) + f'(x_1)(x - x_1) = 0$

Solve for x: $x = x_1 - \dfrac{f(x_1)}{f'(x_1)}$

Use this as next guess, call it x_2. Then $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$

Now continue: $x_3 = x_2 - \dfrac{f(x_2)}{f'(x_2)}$ ... This is a DTDS!

Newton's method

To find a zero of a differentiable function f, take initial guess x_0 and iterate:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Often (but not always) x_n converges to a zero of f.

Example: Calculate $\sqrt{2}$ by solving $f(x) = 0$ with $f(x) = x^2 - 2$

So $f(x) = x^2 - 2$; $f'(x) = 2x$; initial guess: $x_0 = 1$.

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{-1}{2} = 1.5$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.5 - \frac{0.25}{3} = 1.4166$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.4166 - \frac{0.0069}{2.833} = 1.414209$$

$(\sqrt{2} = 1.41421)$

$\Rightarrow$ very fast convergence.

Example: $f(x) = x - e^{-x}$ (from above)

$f'(x) = 1 + e^{-x}$, $x_0 = 0$ initial condition

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{-1}{2} = \frac{1}{2} = 0.5$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = \frac{1}{2} - \frac{\frac{1}{2} - e^{-1/2}}{1 + e^{-1/2}} = \frac{1}{2} + \frac{0.1065}{1.6065} = 0.5663$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = \ldots = 0.5671$$

$x_4 = x_3$ to 4 decimals.

$f(x_4) = -5 \cdot 10^{-15} \simeq 0.$

Antiderivatives

Motivation: Utility companies measure the rate of flow of water / gas / electricity into your home in litres/second ; m^3/second ; watts = joule/second. In the end, however, they bill you for total amount consumed, i.e. litres, m^3, kWh. So, they measure the instantaneous rate of change and calculate the total amount. This is the reverse process of differentiation.

Question and Goal: Do this reverse process mathematically. How?

$f(x)$		$f'(x)$
value	$\xrightarrow{\text{differentiation}}$	rate of change
position		velocity
mass	$\xleftarrow{\text{new}}$	growth rate
volume		flow rate
amount		production rate

So, the goal is: given $f'(x)$ find $f(x)$ or $f'(t)$ find $f(t)$

Note: We often choose t as independent variable if we think of time.

Example: Volume of a cell increases by $2\mu m^3$ per second. What is the volume of the cell after 3 seconds if the cell starts at $1\mu m^3$?

Denote $V(t)$ = volume of cell. Then $V'(t) = 2\mu m^3/s$

Derivative constant $\Rightarrow$ function linear $\Rightarrow$ $V(t) = \frac{2\mu m^3}{s} \cdot t + V(0)$

Initial value $V(0) = 1\mu m^3$. $\Rightarrow V(t) = (2t+1)\mu m^3$.

Note: 1) The units work out!

2) Different initial conditions give different volumes, as it should be. Without initial conditions, we don't know $V(t)$.

Then $V(3) = 7\mu m^3$.

Example: The amount of radioactive material decreases by 1% per day. If there is 1g initially, when will there be 0.5g?

Answer: Write $x(t)$ for the amount. Then $x'(t) = -0.01\,x(t)$

So the derivative is a multiple of the function itself. We know a solution for that: the exponential function:

$x(t) = e^{\alpha t} \Rightarrow x'(t) = \alpha e^{\alpha t} = \alpha x(t)$ so we need to set $\alpha = -0.01$

How about an initial condition as before?

$x(t) = K e^{\alpha t}$ also satisfies $x'(t) = K\alpha e^{\alpha t} = \alpha x(t)$.

Then $x(0) = K$. Here is the initial condition.

$\Rightarrow x(t) = 1g \cdot e^{-0.01 \cdot t}$.

Units? α has units $\frac{\%}{\text{day}} = \frac{1}{\text{day}}$

Then αt has no units, and so $x(t)$ has units g. (grams)

When will $x(t) = 0.5$?

$$0.5 = 1 \cdot e^{-0.01 t} \Rightarrow \ln\left(\tfrac{1}{2}\right) = -0.01\,t \Rightarrow t = \frac{\ln 2}{0.01}$$

Note: Big difference between $V'(t) = 2$ (independent of V)

and $x'(t) = -0.01\,x(t)$ (dependent on x)

1330 We call the first kind pure-time differential equations. The right hand side may depend on t (the independent variable) but not on V (the dependent variable that we are looking for).

1332 The second kind is called autonomous differential equation. The rate of change depends on x (the dependent variable) but not on t (the independent variable).

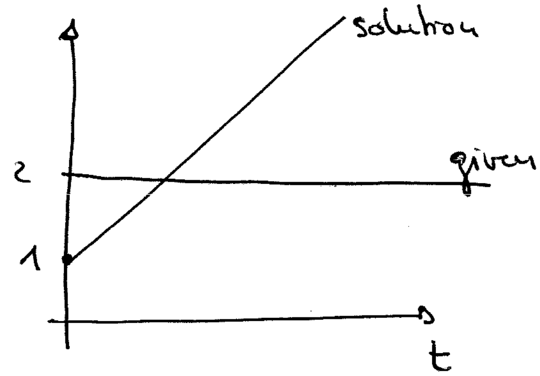
Here, we deal only with pure-time differential equations.

We can try to get solutions by graphing. Given a derivative, find function

Example:

$\frac{dV}{dt} = 2$

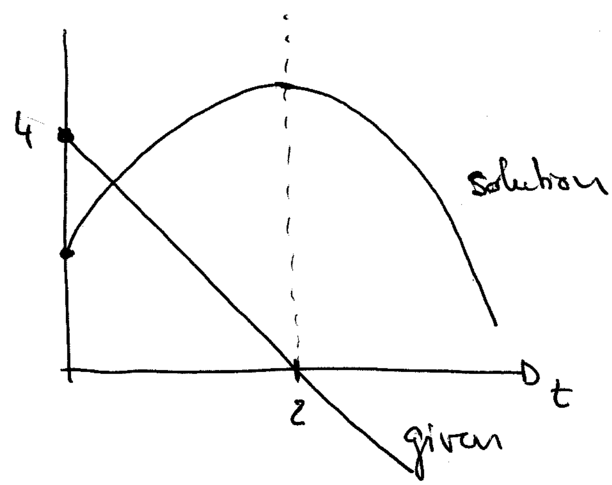
$V(0) = 1$



Example:

$\frac{dG}{dt} = 4 - 2t$

$G(0) = 2$



Autonomous or pure time? Solution?

Example:

a) $x'(t) = e^t$, $x(0) = 1$

b) $y'(t) = \frac{1}{t}$, $y(1) = 0$

c) $w'(t) = 2w(t)$, $w(0) = 1$

d) $z'(t) = \frac{1}{z(t)}$, $z(0) = 1$

Goal for 1330: Find rules and procedures to solve pure-time differential equations (rather than just guessing).

This is reversing the process of differentiation.

So, the goal is: Given $f(t)$, find $F(t)$ such that $F'(t)=f(t)$ | 1330

Let's introduce notation.

<u>Definition</u>: An <u>antiderivative</u> of a function $f(t)$ is a function $F(t)$ with the property $\frac{dF}{dt}=f(t)$. We write

$$F(t)=\int f(t)\,dt$$

and say "integral of f of t dt".

Function f is called the <u>integrand</u>.

We already know: If F is an antiderivative of f, then so is $F+c$.

We write

$$\int f(t)\,dt = F(t)+c$$

and say that $\int f(t)\,dt$ is <u>the indefinite integral of f</u>.

<u>Example</u>: $f(t)=1$. Then $F(t)=t$ is an antiderivative.

$F(t)=t+5$ is also an antiderivative.

And $\int f(t)\,dt=\int 1\,dt = t+c$ with $c\in\mathbb{R}$ is the indefinite integral.

In the following, we will turn all the differentiation rules into rules for antiderivatives. For example:

<u>The power rule</u>: From $\frac{d}{dt}(t^{n+1})=(n+1)t^n$ we get

$$\int t^n\,dt = \frac{1}{n+1}t^{n+1}+c \qquad \text{as long as } n\neq -1$$

<u>Example</u>: $\int t^3\,dt = \frac{1}{4}t^4+c$

$\int x^5\,dx = \frac{1}{6}x^6+c$ Note, if we use a different variable, then dt changes accordingly.

1330 69.5

$$\int \sqrt{t}\, dt = \int t^{1/2} dt = \frac{1}{\frac{3}{2}} t^{3/2} + c = \frac{2}{3} t^{3/2} + c$$

$$\int \frac{1}{x^2} dx = \int x^{-2} dx = \frac{1}{-1} x^{-1} + c = -\frac{1}{x} + c$$

<u>Constant product rule</u>

$$\frac{d}{dx}(af(x)) = a \frac{d}{dx} f \quad \to \quad \int a f(x) dx = a \int f(x) dx + c$$

<u>Sum rule:</u>

$$\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx} f(x) + \frac{d}{dx} g(x) \quad \to \quad \int (f(x) + g(x)) dx = \int f(x) dx + \int g(x) dx$$

<u>Example:</u>

$$\int (3x^2 - 7x) dx = 3 \int x^2 dx - 7 \int x\, dx = 3 \cdot \frac{1}{3} x^3 - 7 \cdot \frac{1}{2} x^2 + c = x^3 - \frac{7}{2} x^2 + c$$

$$\int \frac{3}{\sqrt{x}} - \frac{4}{x^3} dx = 3 \int x^{-1/2} dx - 4 \int x^{-3} dx = 3 \cdot \frac{1}{\frac{1}{2}} x^{1/2} - 4 \cdot \frac{1}{-2} x^{-2} + c$$

$$= 6\sqrt{x} + 2x^{-2} + c.$$

<u>Special functions</u>

$$(\ln(x))' = \frac{1}{x} \quad \to \quad \int \frac{1}{x} dx = \ln|x| + c$$

$$(e^x)' = e^x \quad \to \quad \int e^x dx = e^x + c$$

$$(\sin(x))' = \cos(x) \quad \to \quad \int \cos(x) dx = \sin(x) + c$$

$$(\cos(x))' = -\sin(x) \quad \to \quad \int \sin(x) dx = -\cos(x) + c$$

1330 69.6

<u>Example</u>: In 1981, there were about 340 cases of HIV infection in the US. Since then, the number of cases has grown by about $500t^2$/year where $t=0$ corresponds to 1981.

How many cases are there in 1991?

Answer: $A(t) = \#$ of cases. $A(0) = 340$.

$$A'(t) = 500t^2$$

$$A(t) = \int A'(t)dt = \frac{500t^3}{3} + c$$

$$A(0) = c = 340$$

$$\Rightarrow A(t) = \frac{500t^3}{3} + 340. \qquad A(10) = \frac{500 \cdot 1000}{3} + 340$$

$t=10 \sim 1991$

<u>Example</u>: Rock falls from a bridge. Constant acceleration due to gravity.

$a = -9.8 \frac{m}{s}$. Velocity: $v'(t) = a(t)$

position: $p'(t) = v(t)$

1) Suppose the bridge is 49m high. Initial speed of rock is 0. Find the equation for $p(t)$. Where is the rock after 1 sec? When will it hit the ground? How fast?

Solution: $v(t) = \int a(t)dt = -9.8t + c$

$$v(0) = 0 \Rightarrow c = 0.$$

$$p(t) = \int v(t)\,dt = -\frac{9.8}{2}t^2 + c$$

$$p(0) = 49 = c$$

$$\Rightarrow p(t) = 49 - 4.9t^2$$

Check the units: 49 m. $4.9\frac{m}{s^2} = \frac{9.8 \frac{m}{s^2}}{2}$ so $4.9\frac{m}{s^2} \cdot t^2$ has units m.

$$p(1) = 49 - 4.9 = 44.1 \text{ m}$$

Hit ground $p(t^*) = 0 \Rightarrow 4.9t^{*2} = 49 \Rightarrow t^{*2} = 10 \Rightarrow t^* = (\pm)\sqrt{10} \approx 3.15$

Terminal velocity: $v(\sqrt{10}) = -9.8 \cdot \sqrt{10} \approx -30.99 \frac{m}{s}$.

2) Suppose the rock is thrown up from the bridge with initial speed $10 \frac{m}{s}$. When will it reach its highest point? How high? When will it hit the ground? How fast?

Solution: $v(t) = \int a(t)dt = \int -9.8\, dt = -9.8t + c$

$$v(0) = c = 10$$

$$p(t) = \int v(t)\, dt = \int (-9.8t + 10)dt = -4.9t^2 + 10t + c$$

$$p(0) = 49 = c$$

$$\Rightarrow p(t) = -4.9t^2 + 10t + 49$$

Highest point: $p'(t) = v(t) = 0$. So $-9.8t + 10 = 0$ so $t = \frac{10}{9.8} \approx 1$

Location: $p\left(\frac{10}{9.8}\right) = \ldots \approx 54.102\,m$ (i.e. ~5 m above bridge)

Hit the ground: $p(t^*) = 0$:

$$t^* = \frac{10 \pm \sqrt{100 + 4 \cdot 4.9 \cdot 49}}{9.8} \approx \frac{43}{9.8} \approx 4.3 \text{ seconds}$$

Terminal velocity:

$$v(t^*) \approx -9.8 \cdot 4.3 + 10 = -42.14 + 10 = -32.14 \frac{m}{s}$$

Integration by substitution

So, we know how to integrate $\int e^x \, dx = e^x + C$ because $(e^x)' = e^x$.

But what if we want to integrate $\int e^{3x} dx = ?$ So let's think. Since we only get an exponential function as a derivative when we start with an exponential function, let's try and see: $(e^{3x})' = 3e^{3x}$ by the chain rule.

Now divide by 3 and get $\left(\frac{e^{3x}}{3}\right)' = e^{3x}$. Then we can integrate:

$$\int e^{3x} dx = \int \left(\frac{e^{3x}}{3}\right)' dx = \frac{1}{3} e^{3x} + C$$

This was not too hard to guess. But how about $\int 2x e^{x^2} dx$?

Again, to obtain e^{x^2} when differentiating, we need to start with e^{x^2} already. So let's try: $\left(e^{x^2}\right)' = e^{x^2} \cdot 2x$ by the chain rule.

But this is exactly what we are looking for. So we integrate

$$\int 2x e^{x^2} dx = \int \left(e^{x^2}\right)' dx = e^{x^2} + C$$

In both cases, we needed the chain rule of differentiation to get the solution. Since every differentiation rule can be turned into an integration rule, let's do that for the chain rule in general.

Recall: $\left[f(g(x))\right]' = f'(g(x)) \cdot g'(x)$ or $\frac{d}{dx}(f \circ g) = \frac{df}{dg} \cdot \frac{dg}{dx}$

Then integrating gives:

$$\boxed{\int f'(g(x)) \cdot g'(x) \, dx = f(g(x)) + C}$$

This rule is called integration by substitution.

In the example: $\int 2xe^{x^2}\,dx$, we have

$$g(x) = x^2, \quad g'(x) = 2x, \quad f(g) = e^g \quad \text{so } f(g(x)) = e^{g(x)} = e^{x^2} \text{ and } f'(g) = e^g$$

Then $\int 2xe^{x^2}\,dx = \int g'(x) f'(g(x))\,dx = f(g(x)) + c$

Let's formalize this a bit more so that we don't have to guess that much.

<u>Recipe for substitution</u>

$$\int f'(g(x))\, g'(x)\,dx$$

① Define new variable $y = g(x)$
[Typically innermost function.]

② Differentiate new variable $\frac{dy}{dx} = g'(x)$

$y = g(x)$ | $g'(x)dx = dy$

③ Multiply by dx $\to$ $dy = g'(x)\,dx$

④ Write the entire integral in terms of the new variable

$$= \int f'(y)\,dy$$

⑤ Integrate

$$= f(y) + c$$

⑥ Substitute back $y = g(x)$

$$= f(g(x)) + c$$

The hardest part is recognizing how to choose $y = g(x)$. After that, it is a matter of doing the algebra correctly. And remember:
Never split dx or dy apart. They belong together.
<u>Remember</u>: Always check your result by differentiating!

Example 1

$$\int e^{3x}\,dx = \int e^{y}\cdot\frac{dy}{3} = \frac{1}{3}\int e^{y}\,dy = \frac{1}{3}\left(e^{y}+c\right) = \frac{1}{3}e^{3x}+\tilde{c}$$

$y = 3x$

$dy = 3dx$

$dx = \frac{1}{3}dy$

Example 2

$$\int \frac{1}{1+5x}\,dx = \int \frac{1}{y}\,\frac{dy}{5} = \frac{1}{5}\int \frac{1}{y}\,dy = \frac{1}{5}\left(\ln|y|+c\right) = \frac{1}{5}\ln|1+5x|+\tilde{c}$$

$y = 1+5x$, $dy = 5dx$, $dx = \frac{1}{5}dy$

Example 3

$$\int \cos(2\pi(x-1))\,dx = \int \frac{1}{2\pi}\cos(y)\,dy = \frac{1}{2\pi}\left(\sin(y)+c\right) = \frac{1}{2\pi}\sin(2\pi(x-1))+\tilde{c}$$

$y = 2\pi(x-1)$

$dy = 2\pi\,dx$

$dx = \frac{1}{2\pi}dy$

Example 4

$$\int \cos x\, e^{\sin x}\,dx = \int e^{y}\,dy = e^{y}+c = e^{\sin x}+c$$

$y = \sin x$, $dy = \cos x\,dx$

Example 5

$$\int \frac{e^{-3t}}{(1+e^{-3t})^3}\,dt = -\frac{1}{3}\int \frac{1}{y^3}\,dy = -\frac{1}{3}\cdot\left(-\frac{1}{2}\right)y^{-2}+c = \frac{1}{6y^2}+c$$

$y = 1+e^{-3t}$

$dy = -3e^{-3t}\,dt$ so $-\frac{1}{3}\,dy = e^{-3t}\,dt$

$$= \frac{1}{6(1+e^{-3t})}+c$$

These are all relatively simple examples. The hard part, in general, is to recognize how to choose the new variable, i.e. to recognize, what is $g(x)$ and what is $g'(x)$.

Example 6 (hard)

$$\int \frac{\tan(x)}{\ln(\cos(x))}\,dx$$

first attempt: $g(x) = \cos(x) = y$

$dy = -\sin(x)\,dx$

Since $\tan(x) = \frac{\sin(x)}{\cos(x)}$, we might be lucky.

$$= \int \frac{-\frac{1}{\cos(x)}}{\ln(y)}\,dy = \int \frac{-1}{y\ln(y)}\,dy$$ looks easier, but not easy.

But we recognize that there is $\ln(y)$ and $\frac{1}{y}$ and that $(\ln y)' = \frac{1}{y}$

So, let's substitute again: $u = \ln(y)$ $du = \frac{1}{y}\,dy$

Then

$$= \int \frac{-1}{u}\,du = -\ln|u|+c = -\ln|\ln(y)|+c = -\ln|\ln(\cos x)|+c$$

Better check:

$$\frac{d}{dx}\left(-\ln|\ln(\cos x)|\right) = -\frac{\frac{d}{dx}\ln(\cos(x))}{\ln(\cos(x))} = -\frac{\frac{-\sin(x)}{\cos(x)}}{\ln(\cos(x))} = \frac{\tan(x)}{\ln(\cos(x))} \checkmark$$

For practice, try directly substituting $y = \ln(\cos(x))$.

This new technique also works in applications.

Example 7:

A fish grows in length, $L(t)$, according to $\frac{dL}{dt} = 7e^{-0.1t}$. L is in cm.

Suppose that $L(0)=0$, how long does it take the fish to reach 50 cm?

Solution: $\frac{dL}{dt} = 7e^{-0.1t} \Rightarrow L(t) = \int 7e^{-0.1t}\,dt$

$$= 7\int (-10)e^{y}\,dy$$

$y = -0.1t$

$$= -70(e^{y}+c)$$

$dy = -0.1\,dt$

$dt = -10\,dy$

$$= -70(e^{-0.1t}+c)$$

$$L(0) = -70(1+c) \Rightarrow c = -1$$

Note: c is absolutely crucial here!

Then $L(t) = -70(e^{-0.1t}-1) = 70(1-e^{-0.1t})$

So $L(t) = 50$ if $1 - e^{-0.1t} = \frac{5}{7}$

$$1 - \frac{5}{7} = e^{-0.1t}$$

$$\frac{2}{7} = e^{-0.1t} \quad \ldots \quad t = \frac{\ln(2/7)}{-0.1}$$

But not everything that looks like a substitution integral is one!

Example 8:

$$\int \frac{(4t+2)^2}{t^2}\,dt = \int \frac{16t^2+16t+4}{t^2}\,dt = \int\left(16 + \frac{16}{t} + \frac{4}{t^2}\right)dt$$

$$= 16t + 16\ln|t| - \frac{4}{t} + c$$

power rule all along

And even with substitution, we still cannot solve all integrals.

Example 9:

$\int \frac{e^{3t}}{(1+e^{-3t})^3}\,dt$ looks very similar to Example 5, but it is not.

Substituting $y = 1+e^{-3t}$ as in Example 5 gives $\int \frac{e^{6t}}{3y^3}\,dy$.

But this integral contains t and y. This cannot work.

We can eliminate t by solving $y = 1+e^{-3t}$

$$y-1 = e^{-3t}, \quad e^{3t} = \frac{1}{y-1}$$

$$\Rightarrow e^{6t} = \frac{1}{(y-1)^2}$$

Then the integral becomes $\int \frac{1}{3y^2(y-1)^2}\,dy$

We only learn how to do integrals of this kind in MAT 1332.

Example 10:

Even worse is the integral $\int 2x^2 e^{x^2}\,dx$. Again, setting $y = x^2$ and $dy = 2x\,dx$, we get $= \int x e^y\,dy = \int \sqrt{y}\, e^y\,dy$

This integral does not have a closed-form solution at all, i.e. we cannot write down an explicit solution. Nobody can.

Integration by parts

When we integrate by substitution, we are looking for a very particular form of product to integrate: $\int f'(g(x))\,g'(x)\,dx$. This comes from the chain rule. But what about the product rule for differentiation, what kind of integration rule do we get from it?

Recall: $(uv)' = u'v + uv'$

Integrate: $\int (uv)'dx = \int u'v\,dx + \int uv'\,dx$ and $\int (uv)'dx = u(x)v(x) + c$

So then:

$$\boxed{\int u'(x)v(x)\,dx = u(x)v(x) - \int u(x)v'(x)\,dx}$$

This rule is called integration by parts. It may look strange at first, because we simply replace one integral with another, but sometimes, it really becomes easier. The hard part, again is to choose $u'(x)$ and $v(x)$ in a clever way.

Example 1:

$$\int xe^x\,dx = x\cdot e^x - \int e^x\,dx = x\cdot e^x - e^x + c = (x-1)e^x + c$$

$\hookrightarrow u'(x) = e^x, \quad u(x) = e^x$

$v(x) = x, \quad v'(x) = 1$

Check!

Example 2:

apply again!

$$\int x^2 e^{3x}\,dx = x^2\frac{e^{3x}}{3} - \int 2x\frac{e^{3x}}{3}\,dx = \frac{x^2e^{3x}}{3} - \frac{2}{3}\int xe^{3x}\,dx$$

$\hookrightarrow u'(x) = e^{3x}, \quad u(x) = \frac{1}{3}e^{3x}$

$v(x) = x^2, \quad v'(x) = 2x$

$\hookrightarrow u'(x) = e^{3x}$

$v(x) = x$

$$= \frac{x^2e^{3x}}{3} - \frac{2}{9}xe^{3x} + \frac{2}{27}e^{3x} + c$$

Example 3:

$$\int \ln(x)\,dx = \int 1\cdot \ln(x)\,dx = x\ln(x) - \int x\cdot \tfrac{1}{x}\,dx = x\ln x - x + c$$

$\hookrightarrow u'(x) = 1,\ u(x) = x$

$\hookrightarrow v(x) = \ln x,\ v'(x) = \frac{1}{x}$

Example 4

$\int x\ln(x)\,dx$ Since we just learned what $\int \ln(x)\,dx$ is, we are tempted to set $u'(x) = \ln(x)$ and $v(x) = x$. But that won't work. Try it out.

Instead:

$$\int x\ln(x)\,dx = \frac{x^2}{2}\ln(x) - \int \frac{x^2}{2}\cdot\frac{1}{x}\,dx = \frac{x^2}{2}\ln(x) - \int \frac{x}{2}\,dx$$

$\hookrightarrow u'(x) = x,\ u(x) = \frac{x^2}{2}$

$v(x) = \ln x,\ v'(x) = \frac{1}{x}$

$$= \frac{x^2}{2}\ln(x) - \frac{x^2}{4} + c$$

Example 5

$$\int \frac{\ln(x)}{x^2}\,dx = -\frac{1}{x}\ln(x) + \int \frac{1}{x^2}\,dx = -\frac{1}{x}\ln(x) - \frac{1}{x} + c$$

$\hookrightarrow u'(x) = \frac{1}{x^2} \to u(x) = -\frac{1}{x}$

$v(x) = \ln x \to v'(x) = \frac{1}{x}$

check!

Example 6: (Try this example by substitution!)

$$\int \frac{\ln(x)}{x}\,dx = \ln(x)\cdot\ln(x) - \int \frac{\ln x}{x}\,dx$$? Same integral again?

$\hookrightarrow u'(x) = \frac{1}{x} \to u(x) = \ln x$

$v(x) = \ln(x) \to v'(x) = \frac{1}{x}$

$$\Rightarrow 2\int \frac{\ln(x)}{x}\,dx = (\ln(x))^2$$

$$\Rightarrow \int \frac{\ln(x)}{x}\,dx = \frac{1}{2}(\ln(x))^2 + c$$

Example 7

The mass of a worm, $\Pi(t)$, changes according to $\Pi'(t) = ate^{-t}$, $a>0$.
Find $\Pi(t)$ and $\lim\limits_{t\to\infty} \Pi(t)$, when $\Pi(0)=0$.

Solution: $\Pi(t) = \int \Pi'(t)\,dt = \int ate^{-t}dt = -ate^{-t} + a\int e^{-t}dt$

$\quad \hookrightarrow u' = e^{-t}, \; u = -e^{-t}$

$\quad\; v = t, \; v' = 1$

$= -ate^{-t} - ae^{-t} + c$

$= -a(t+1)e^{-t} + c$

From $\Pi(0)=0$, we find $c=a$.

Then $\Pi(t) = a\left(1 - (1+t)e^{-t}\right)$

Now, $\lim\limits_{t\to\infty} \dfrac{1+t}{e^t} \underset{L'H}{=} \lim\limits_{t\to\infty} \dfrac{1}{e^t} = 0$, so $\lim\limits_{t\to\infty} \Pi(t) = a$.

Example 8: Not all things work with integration by parts:

$\int e^x \cdot \frac{1}{x}\,dx = ?$ 1) $\left.\begin{array}{l} u' = e^x, \; u = e^x \\ v = \frac{1}{x}, \; v' = -\frac{1}{x^2} \end{array}\right\} \Rightarrow = \dfrac{e^x}{x} - \int \dfrac{e^x}{x^2}dx$

This is worse.

2) $\left.\begin{array}{l} u'(x) = \frac{1}{x}, \; u(x) = \ln x \\ v(x) = e^x, \; v'(x) = e^x \end{array}\right\} \Rightarrow = e^x \ln|x| - \int e^x \ln(x)\,dx$.

not any better.

Example 9:

$$\int e^x \sin(x)\,dx = e^x \sin(x) - \int e^x \cos(x)\,dx = e^x \sin(x) - e^x\cos(x) + \int e^x(-\sin x)\,dx$$

$\hookrightarrow u'(x) = e^x,\ u(x) = e^x$

$v(x) = \sin(x),\ v'(x) = \cos(x)$

$\hookrightarrow u' = e^x,\ u = e^x$

$v(x) = \cos x,\ v' = -\sin x$

Again, the first and the last integral are the same, so

$$2\int e^x \sin(x)\,dx = e^x\big(\sin(x) - \cos(x)\big) + c$$

or $$\int e^x \sin(x)\,dx = \frac{1}{2} e^x\big(\sin(x) - \cos(x)\big) + c$$

Example 10:

$$\int \sin(\sqrt{x})\,dx = ?$$

First substitution: $y = \sqrt{x},\ dy = \frac{1}{2\sqrt{x}}\,dx$ or $dx = 2\sqrt{x}\,dy = 2y\,dy$

$$= \int 2y \sin(y)\,dy = -2y\cos(y) + 2\int \cos(y)\,dy = -2y\cos(y) + 2\sin(y) + c$$

$\hookrightarrow u' = \sin(y),\ u = -\cos(y)$

$v = y,\ v' = 1$

$$= 2\sqrt{x}\cos(\sqrt{x}) + 2\sin(\sqrt{x}) + c$$

check this result by differentiating!

Practice Problem

Consider the DTDS $x_{t+1} = (1-h)\cdot 2x_t(1-x_t)$

where x_t is the population density at time t and h is the fraction harvested, i.e. $(1-h)$ the fraction remaining. We require $0 \leq h < 1$.

1) Find the expression $x^* = x^*(h)$ of the positive steady state as a function of harvesting rate. What is the maximal harvesting rate for which x^* is positive?

2) Find h that maximizes the yield $y = h \cdot x^*$.

Solution

1) Steady state: $x = 2(1-h)\,x(1-x)$ solve for x. (Not h!)

$$x = 0 \quad \text{or}$$

$$1 = 2(1-h)(1-x)$$

$$\Leftrightarrow \quad \frac{1}{2(1-h)} = 1 - x$$

$$\Leftrightarrow \quad x^* = 1 - \frac{1}{2(1-h)} = \frac{1-2h}{2-2h}$$

2) Positivity of x^*:

$x^* > 0$ if $1-2h > 0$ and $2-2h > 0$ or if both are negative.

Since $0 \leq h < 1$, the denominator is always positive. So, the condition is $1-2h > 0$ or $\boxed{h < \frac{1}{2}}$

3) Maximize $y = h \cdot x^* = \dfrac{h(1-2h)}{2-2h}$

Differentiate: $y'(h) = \dfrac{(1-4h)(2-2h) - (h-2h^2)(-2)}{(2-2h)^2} = \dfrac{1-4h+2h^2}{2(1-h)^2}$

$$y'(h) = 0 \quad \Leftrightarrow \quad 2h^2 - 4h + 1 = 0 \quad \Leftrightarrow \quad h = 1 \pm \sqrt{\tfrac{1}{2}}$$

- Since we look at $0 < h < 1$, only $h = 1 - \sqrt{\frac{1}{2}} \approx 0.29$ is admissible
- Since $y = hx^* > 0$ between $h=0$ and $h = \frac{1}{2}$ and $hx^* = 0$ for $h=0$ and $h = \frac{1}{2}$, it must have a maximum for $h \in (0, \frac{1}{2})$. Since there is only one extremum at $h = 1 - \sqrt{\frac{1}{2}}$, it has to be a max.
- Note that $h^* = 1 - \sqrt{\frac{1}{2}} < \frac{1}{2}$, so the optimal harvesting rate is feasible.