

Practice Final Exam for MAT 2377

Only faculty standard calculators are permitted. It is a closed book exam, but two sheets (double sided) are permitted. There are 3 short answer questions and 15 multiple choice questions.

Short Answer Questions

1. There is a chance that a bit transmitted through a digital transmission channel is received in error. Let X equal the number of bits in error in the next four bits transmitted. Suppose that X has the following probability mass function.

$$P(X = x) = \begin{cases} 0, & 0.66 \\ 1, & 0.29 \\ 2, & 0.04 \\ 3, & 0.009 \\ 4, & 0.001 \end{cases}$$

- (a) What is the probability that there is at most 1 bit in error in the next four transmitted bits?
(b) What is the expected number of bits in error in the next four bits transmitted?
(c) Compute the standard deviation of the number of bits in error in the next four bits transmitted.
(d) Compute $P\left(-1.96 < \frac{X - E[X]}{\sqrt{V[X]}} < 1.96\right)$.

Solution: (a) $P(X \leq 1) = P(X = 0) + P(X = 1) = 0.95$.

(b) $E[X] = (0)(0.66) + 1(0.29) + 2(0.04) + 3(0.009) + 4(0.001) = 0.401$

(c) $V[X] = 0^2(0.66) + 1^2(0.29) + 2^2(0.04) + 3^2(0.009) + 4^2(0.001) - (0.401)^2 = 0.386199$.
Then, $\sigma_X = \sqrt{V[X]} = 0.62145$.

(d)

$$\begin{aligned} & P\left(-1.96 < \frac{X - E[X]}{\sqrt{V[X]}} < 1.96\right) \\ &= P\left(-1.96 < \frac{X - 0.401}{0.62145} < 1.96\right) \\ &= P(-1.96(0.62145) + 0.401 < X < 1.96(0.62145) + 0.401) \\ &= P(-0.8170 < X < 1.619) \\ &= P(X = 0) + P(X = 1) = 0.95 \end{aligned}$$

2. A manufacturer wants to test the lifetime of a small motor it builds; the mean lifetime is supposed to be at least 4 years. Assume that the distribution of lifetimes is approximately normal.
- (a) The manufacturing unit will do a thorough investigation of the manufacturing process if it has reason to believe that the lifetime is too short. Formulate a null and alternative hypothesis.
- (b) The CEO has called and said that selling substandard motors is unacceptable. Now an investigation will be done unless there is evidence that the mean lifetime meets requirements. Formulate a null and alternative hypothesis.
- (c) A random sample of 15 motors is made. Under the situation of **part (b)**, and with a significance of 0.05, formulate the test procedure.
- (d) Suppose that the sample mean is 4.13 years and the sample standard deviation is 0.35 years. What does the test procedure in part (c) say to do?

Solution: Let μ be the mean lifetime in years.

(a) We want to test $H_0 : \mu = 4$ against $H_1 : \mu < 4$.

(b) We want to test $H_0 : \mu = 4$ against $H_1 : \mu > 4$.

(c) It is a right-sided alternative, so we will reject H_0 if $t_0 > t_{0.05;14} = 1.761$, where $t_0 = (\bar{x} - 4)/(s/\sqrt{15})$.

(d) Since $t_0 = (\bar{x} - 4)/(s/\sqrt{15}) = (4.13 - 4)/(0.35/\sqrt{15}) = 1.438$, then $t_0 < 1.761$. The evidence against H_0 is not significant. The investigation will proceed.

3. Let X be the number of soldering defects for a device. It has the following mean $\mu = 1.2$ defects and standard deviation $\sigma = 1.6$ defects. Furthermore, the probability that a device will have more than 2 defects is 0.2.
- (a) Suppose that we ship 20 devices what is the probability that at most one device will have more than 2 defects?
- (b) If we ship the devices one by one, what is the probability that the sixth device is the first device with more than 2 defects?
- (c) Each soldering defect takes about 10 minutes to fix. If we ship 50 devices to the customer, approximate the probability a repair man would take more than 13.75 hours to fix all of the defects.

solution:

- (a) Let X be the number of devices among $n = 20$ with more than 2 defects. X has a binomial distribution with $n = 20$ and $p = 0.2$. We want

$$P(X \leq 1) = \binom{20}{0} p^0 (1-p)^{20} + \binom{20}{1} p^1 (1-p)^{19} = 0.0692.$$

- (b) Let Y be the number of devices required to observe a device with more than 2 defects. Y has a geometric distribution with $p = 0.2$. We want

$$P(Y = 6) = (1 - p)^5 p = 0.0655.$$

- (c) Let T_i be the time (in minutes) to fix the defects of the i th device. It is $T_i = 10 X_i$, where X_i is the number of defects for the i th device. We have

$$\mu = E[T_i] = 10 E[X_i] = 10(1.2) = 12$$

and

$$\sigma = \sqrt{V[T_i]} = \sqrt{10^2 V[X_i]} = \sqrt{100(1.6)} = 12.6491.$$

The probability that it would take more than 13.75 hours (or equivalently 825 minutes) to fix all of the defects for 50 devices is

$$\begin{aligned} P\left(\sum_{i=1}^{50} T_i > 825\right) &= P\left(\sum_{i=1}^{50} T_i/50 > 825/50\right) \\ &= P(\bar{T} > 16.5) \\ &\approx 1 - \Phi\left(\frac{16.5 - 12}{12.6491/\sqrt{50}}\right) \quad (\text{by the CLT}) \\ &= 1 - \Phi(2.52) \\ &= 1 - 0.9941 = 0.0059. \end{aligned}$$

Multiple choice questions

1. Printed circuit cards are placed in a functional test after being populated with semiconductor chips. A lot contains 140 cards, and 20 are selected without replacement for functional testing. If 6 of the 140 cards are defective, what is the probability that at most 1 defective card will appear in the sample.

A) 0.4062 B) 0.7956 C) 0.01578 D) 0.9754 E) 0.2065

solution: Let X be the number of defective cards in the sample. We want

$$\begin{aligned} P(X \leq 1) &= P(X = 0) + P(X = 1) \\ &= \frac{\binom{134}{20} \binom{6}{0}}{\binom{140}{20}} + \frac{\binom{134}{19} \binom{6}{1}}{\binom{140}{20}} \\ &= 0.7956 \end{aligned}$$

The answer is B.

2. A wire manufacturer wants to ensure that the mean radius of one model is very close to its listed value. Suppose that the standard deviation of the radius is known to be $10 \mu\text{m}$. The population mean will be approximated by the average radius from a random sample of manufactured wires. How many wires need to be sampled to have 95% confidence that the error is at most $3 \mu\text{m}$?

A) 30 B) 31 C) 42 D) 43 E) 53

solution: Solve

$$n \geq \left[\frac{z_{0,025} \sigma}{E} \right]^2 = \left[\frac{(1,96)(10)}{3} \right]^2 = 42,684.$$

We will select $n = 43$ wires. The answer is D.

3. Calls to the help line of a large computer distributor occur according to a Poisson process with a rate of 20 calls per minute. What is the probability that three or more calls occur within 15 seconds?

A) 0.1246 B) 0.8753 C) 0.7350 D) 0.2650 E) 0.4405

solution: We have a Poisson process with a rate of $\lambda = 20$ calls per minute. Let $N(1/4)$ be the number of calls in 15 seconds (that is $1/4$ minute). $N(1/4)$ has a Poisson distribution with mean $\mu = \lambda t = (20)(1/4) = 5$. We want

$$P(N(1/4) \geq 3) = 1 - \left[e^{-5} \frac{e^0}{0!} + e^{-5} \frac{e^1}{1!} + e^{-5} \frac{e^2}{2!} \right] = 0.8753.$$

The answer is B.

4. Suppose that the joint probability mass function of the random variables X and Y is given by the following table:

x	y	$p_{XY}(x, y)$
0	0	1/12
0	2	3/12
1	1	1/12
1	2	4/12
2	0	2/12
2	2	1/12

Compute $P(X + Y \geq 2)$.

- A) 2/12 B) 7/12 C) 6/12 D) 9/12 E) 11/12

solution:

$$\begin{aligned} P(X + Y \geq 2) &= p_{X,Y}(0, 2) + p_{X,Y}(1, 1) + p_{X,Y}(1, 2) + p_{X,Y}(2, 0) + p_{X,Y}(2, 2) \\ &= \frac{3}{12} + \frac{1}{12} + \frac{4}{12} + \frac{2}{12} + \frac{1}{12} = \frac{11}{12} \end{aligned}$$

The answer is E.

5. Suppose the cumulative distribution function of a continuous random variable X is

$$F(x) = \begin{cases} 0, & x \leq 0 \\ 0.25(x - 2) + 0.5, & 0 \leq x \leq 4 \\ 1, & x \geq 4 \end{cases}$$

Compute the following expectation $E[X]$.

- A) 1.4142 B) 0.3333 C) 1.5333 D) 2 E) 1.3333

solution: X has the following probability density function.

$$f(x) = F'(x) = 0.25, \quad 0 < x < 4.$$

So its mean is

$$E[X] = \int_{-\infty}^{\infty} x f(x) dx = \int_0^4 x (0.25) dx = 0.25 \left[\frac{4^2}{2} - \frac{0^2}{2} \right] = 2.$$

The answer is D.

6. The strength of a steel alloy is normally distributed with a mean of 20 gigapascals (GPa) and a standard deviation of 3.54 GPa. What is the probability that a specimen of this alloy will have a strength at least 24 GPa?

- A) 0.1151 B) 0.1292 C) 0.3206 D) 0.3409 E) 0.8849

solution: We want

$$P(X \geq 24) = 1 - \Phi\left(\frac{24 - 20}{3.54}\right) = 1 - \Phi(1.13) = 0.1292.$$

The answer is B.

7. Each sample of water has a 10% chance of containing a particular organic pollutant. Assume that the samples are independent with regard to the presence of the pollutant. Find the probability that in the next 16 samples, exactly two contain the pollutant.

A) 0.3245 B) < 0.0001 C) 0.2745 D) 0.0065 E) 0.7892

Solution: Let X be the number of samples among the $n = 16$ samples that contain the pollutant. X has a binomial with $n = 16$ and $p = 0.1$. We want

$$P(X = 2) = \binom{16}{2} (0.1)^2 (0.9)^{14} = 0.2745.$$

The answer is C.

8. A sample of 17 randomly selected boxes of a particular brand of cereal is taken, and each box is weighed. The sample mean is calculated to be 460 g, and the sample standard variation is calculated to be 23 g. Find the number x so that with probability 90%, the true population mean is at most x . Assume that the weights are normally distributed.

A) 1.333 B) 452.564 C) 467.140 D) 467.436 E) 467.458

solution: We are 90% confident that

$$\mu < \bar{x} + t_{0.1,16} \frac{s}{\sqrt{n}} = 460 + (1.337) \frac{23}{\sqrt{17}} = 467.458.$$

The answer is E.

9. An engineering firm has a Facebook page. The visits to the page occur according to a Poisson process with a rate of 4 visits per day. What is the probability that we need to wait more than one day to observe 4 visits?

A) 0.0915 B) 0.4335 C) 0.6288 D) 0.0183 E) 0.2381

solution: Let $N(1)$ be the number of visits in a day. $N(1)$ has a Poisson distribution with mean $\mu = \lambda t = 4(1) = 4$. We want

$$P[N(1) \leq 3] = e^{-4} \frac{4^0}{0!} + e^{-4} \frac{4^1}{1!} + e^{-4} \frac{4^2}{2!} + e^{-4} \frac{4^3}{3!} = 0.4335.$$

The answer is B.

10. Let X_1, \dots, X_{20} be a random sample from a normal population with mean $\mu = 16$ and standard deviation $\sigma = 3$. Write \bar{X} for the sample mean. Find the number c so that $P(\bar{X} < c) = 0.95$.

A) 16.556 B) 16.859 C) 17.100 D) 17.160 E) 20.920

solution: $Z = (\bar{X} - 16)/(3/\sqrt{20})$ has a standard normal distribution. We want c such that

$$0.95 = P(\bar{X} < c) = \Phi \left(\frac{c - 16}{3/\sqrt{20}} \right).$$

Thus,

$$1.645 = \frac{c - 16}{3/\sqrt{20}} \Rightarrow c = (1,645)(3/\sqrt{20}) + 16 = 17.1035 \approx 17.10.$$

The answer is C.

11. Engineers of a fleet must continually check for corrosion inside the pipes that are part of the cooling systems. The inside condition of the pipes cannot be observed directly but a nondestructive test can give an indication of possible corrosion. The test is not perfect. The test has a probability of 0.7 of detecting internal corrosion when internal corrosion is present but it also has a probability of 0.1 of detecting internal corrosion, when there is no internal corrosion. Suppose the probability that any section of pipe has internal corrosion is 0.15. Given that the test detects internal corrosion, what is the probability that there truly is internal corrosion?

A) 0.2916 B) 0.3600 C) 0.1500 D) 0.3425 E) 0.1926

Solution: Let I be the event that there truly is internal corrosion that is present and D that the test detects internal corrosion. We have

$$P(I) = 0.15; \quad P(D|I) = 0.7; \quad P(D|I') = 0.3.$$

We want

$$\begin{aligned} P(I|D) &= \frac{P(I \cap D)}{P(D)} = \frac{P(I \cap D)}{P(D)} \\ &= \frac{P(D|I)P(I)}{P(D|I)P(I) + P(D|I')P(I')} \\ &= \frac{(0.7)(0.15)}{(0.7)(0.15) + (0.3)(0.85)} \\ &= 0.2916. \end{aligned}$$

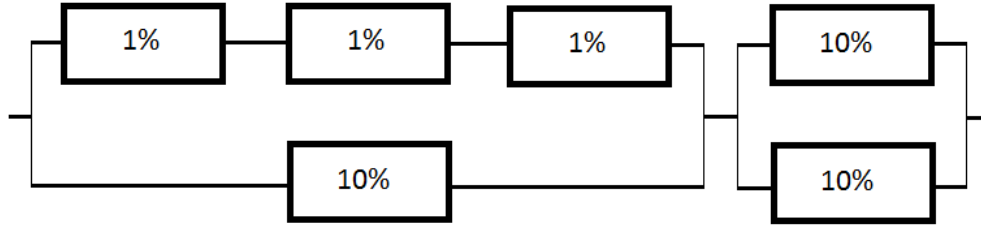
12. Consider a random sample of size $n = 22$ from a normal population with a mean of 15. Let \bar{X} and S be the sample mean and the sample standard deviation, respectively. Find a such that

$$P\left(\frac{\bar{X} - 5}{S/\sqrt{22}} \leq a\right) = 0.01.$$

A) 2.518 B) -2.518 C) -2.831 D) 2.508 E) -2.508

solution: The random variable $T = (\bar{X} - 5)/(S/\sqrt{22})$ has a $T(21)$ distribution. We want $a = t_{0.99;21} = -t_{0.01;21} = -2.518$. The answer is B.

13. The following circuit operates if and only if there is a path of functional devices from left to right. Assume devices fail independently and that the probability of *failure* of each device is as shown. What is the probability that the circuit operates?



- A) 0.9871 B) 0.9975 C) 0.9725 D) 0.0211 E) 0.7885

The first sub-circuit A works (on the left), if its top circuit A_1 works or its bottom circuit A_2 works. We have

$$P(A_1 \cup A_2) = 1 - P(A_1')P(A_2') = 1 - (0.0297)(0.1) = 0.99703,$$

where

$$P(A_1) = 1 - (0.01)^3 = 0.9703 \quad \text{and} \quad P(A_2) = 0.9.$$

The second sub-circuit B (on the right) works if at least one of the two components work. Thus, $P(B) = 1 - (0.1)^2 = 0.99$. The circuit works if A and B work, so

$$P(A \cap B) = P(A)P(B) = (0.99703)(0.99) = 0.9871.$$

The answer is A.

14. Let X equal the grade of a student in a statistics course. Suppose that X has a normal distribution. Using traditional method of teaching, the average student grade was $\mu = 60$. We want to test if the the students' grades in this course have increased on average due to a new teaching method. In order to test $H_0 : \mu = 60$ versus $H_1 : \mu > 60$, we will use a random sample of $n = 26$ students. We computed $\bar{x} = 62.5$ and $s = 10$ for the 26 students. Find the p -value for this test.

- A) p -value > 0.10
 B) $0.05 < p$ -value < 0.10
 C) $0.025 < p$ -value < 0.05
 D) $0.01 < p$ -value < 0.025
 E) p -value < 0.01

solution: The observed value of the t -test statistic is

$$t_0 = \frac{\bar{x} - 60}{s/\sqrt{26}} = \frac{62.5 - 60}{10/\sqrt{26}} = 1.275.$$

Since it is a right-sided alternative, then the p -value is $P(T > 1.275)$, where T has a $T(25)$ distribution. Since $t_{0.25,25} = 0.684$ and $t_{0.10,25} = 1.316$, Then, $0.10 < p$ -value < 0.25 . Then, p -value > 0.10 .

15. The determination of the shear strength of spot welds is relatively difficult, whereas measuring the weld diameter of spot welds is relatively simple. We will describe the shear strength (in psi) as a linear function of the weld diameter (in 0.0001 inch). The data are as follows:

i	Shear Strength y_i (psi)	Weld diameter x_i (0.0001 inch)
1	370	400
2	780	800
3	1210	1250
4	1560	1600
5	1980	2000
6	2450	2500
7	3070	3100
8	3550	3600
9	3940	4000
10	3950	4000
total	22,860.0	23,250.0

From these data, we computed the following quadratic forms:

$$\sum_{i=1}^{10} (x_i - \bar{x})^2 = 15,686,250.0; \quad \sum_{i=1}^{10} (y_i - \bar{y})^2 = 15,461,440.0;$$

$$\sum_{i=1}^{10} (x_i - \bar{x})(y_i - \bar{y}) = 15,573,000.0.$$

Assuming that it is reasonable to use a simple linear regression model to describe the shear strength as a function of the weld diameter, predict the shear strength in (psi) of a spot weld when the weld diameter is 0.375 inch.

- A) 3701 B) 3850 C) 3576 D) 3695 E) 3687

Solution: The slope is

$$\hat{\beta}_1 = \frac{s_{xy}}{s_{xx}} = \frac{15,573,000.0}{15,686,250.0} = 0.99278$$

and the intercept is

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x} = \frac{22,860.0}{10} - 0.99278 \left(\frac{23,250.0}{10} \right) = -22.2135.$$

The estimated regression line is

$$\hat{y} = -22.2135 + 0.99278 x.$$

Note that a diameter of 0.375 inch is $x = 3750$ (in 0.0001 in). The predicted value is

$$\hat{y} = -22.2135 + 0.99278 (3750) = 3700.712 \approx 3701 \text{ psi.}$$