MAT 2377 - Winter 2017
Practice Exam 2 - Solutions

1. $X = \#$ of failures $\sim B(20, 0.09)$. We want

$$\begin{aligned} P(X \leq 2) = & \binom{20}{0} (.09)^0 (.91)^{20} + \binom{20}{1} (.09)^1 (.91)^{19} \\ & + \binom{20}{2} (.09)^2 (.91)^{18} \\ = & 0.73 \end{aligned}$$

2. $X = \#$ of required mice. $X \sim$ geometric with $p = 1/6$. We want $P(X = 8) = (1 - p)^7\, p = 0.0465$

3. Let A=positive result and M=having the disease. We know that $P(A|M) = \frac{436}{450}$, $P(A|M') = \frac{5}{500}$, $P(M) = 0.077$. Using Bayes' Formula :

$$P(M|A) = \frac{P(A|M)P(M)}{P(A|M)P(M) + P(A|M')P(M')} = 0.89$$

4. Let A_i= ith component works. The probability that the circuit is working is

$$\begin{aligned} & P(A_1 \cap (A_4 \cup (A_2 \cap A_3))) \\ = \;& P(A_1)P(A_4 \cup (A_2 \cap A_3)) \quad \text{(by independence)} \\ = \;& P(A_1)[P(A_4) + P(A_2 \cap A_3) - P(A_4 \cap A_2 \cap A_3)] \quad \text{(addition rule)} \\ = \;& P(A_1)[P(A_4) + P(A_2)P(A_3) - P(A_4)P(A_2)P(A_3)] \quad \text{(by independence)} \\ = \;& 0.5\,[0.5 + (0.4)(0.7) - (0.5)(0.4)(0.7)] = 0.32 \end{aligned}$$

Thus, the probability that the circuit is not working is $1 - 0.32 = 0.68$.

5.

$$\mu_X = E[X] = \sum_{x \in R_X} x\, f_X(x) = 1.6$$

6.

$$E[X^2] = \int_0^1 x^2\,(3\,x^2)\,dx = 0.6 \;\Rightarrow\; \sigma_X^2 = E[X^2] - \mu_X^2 = 0.0375$$

7. This is probability density function for an exponential random variable with $\lambda = 1/2$. Thus, the mean and the variance of the population are : $\mu = 1/\lambda = 2$, $\sigma^2 = (1/\lambda)^2 = 4$. Hence,

$$\begin{aligned} P\left(\sum_{i=1}^{81} X_i > 170\right) &= P\left(\overline{X} > 170/81\right) \approx 1 - \Phi\left(\frac{170/81 - 2}{\sqrt{4}/\sqrt{80}}\right) \\ &= 1 - \Phi(0.44) = 0.33 \end{aligned}$$

8. $\overline{X}_1 + \overline{X}_2 \sim N(75 + 80, 288/16 + 162/9) = N(155, 36)$. Therefore,

$$\begin{aligned} P(\overline{X}_1 + \overline{X}_2 > 156.5) &= 1 - \Phi\left(\frac{156.5 - 155}{\sqrt{36}}\right) \\ &= 1 - \Phi\,(0.25) = 0.4013 \end{aligned}$$

9. Since

$$V[\widehat{\Theta}_1] = \left(\frac{1}{2}\right)^2 \sigma^2 + \left(\frac{1}{2}\right)^2 \sigma^2 = \frac{\sigma^2}{2}$$

and

$$V[\widehat{\Theta}_2] = 2^2\,\sigma^2 + (-1)^2\,\sigma^2 = 5\,\sigma^2,$$

therefore, $V[\widehat{\Theta}_1] < V[\widehat{\Theta}_2]$.

10. The sample mean and sample standard deviation are

$$\overline{x} = 1.009 \quad \text{and} \quad s = \sqrt{\frac{\sum_{i=1}^{n} x_i^2 - n\,\overline{x}^2}{n-1}} = 0.02558.$$

Note that $t_{.005,10-1} = 3.250$. A 99% confidence interval for μ is

$$\overline{x} \pm 3.250\,\frac{s}{\sqrt{n}} = [0.983, 1.035].$$

11. $X \sim N(31, 0.2^2)$. We want

$$\begin{aligned} P(30.5 < X < 31.5) &= \Phi\left(\frac{31.5 - 31}{0.2}\right) - \Phi\left(\frac{31.5 - 31}{0.2}\right) \\ &= \Phi(2.5) - \Phi(-2.5) = 0.9876 \end{aligned}$$

12. *Conditions :* normal population with σ known.
An 95% confidence interval for μ is

$$\overline{x} \pm z_{.025}\frac{\sigma}{\sqrt{n}} = 98 \pm 1.96\left(\frac{\sqrt{2}}{\sqrt{25}}\right) = [97.446, 98.554].$$

13. The lower fence is $q_1 - 1.5\,\text{IQR} = 9.825 - 1.5(10.280 - 9.825) = 9.1425$. The upper fence is $q_3 + 1.5\,\text{IQR} = 10.280 + 1.5(10.280 - 9.825) = 10.9625$. There are two values that fall outside of the fences : 8.7 and 8.8. Thus, 8.7 and 8.8 are the only outliers.
14. The observed value of the test statistic is

$$z_0 = \frac{\overline{x} - 95}{\sigma/\sqrt{n}} = \frac{94.32 - 95}{1.2/\sqrt{16}} = -2.27.$$

It is a two-sided test, hence
$P = 2P(Z > |-2.27|) = 2\left(1 - \Phi(2.27)\right) = 0.023$. Since $P > \alpha$, thus we do not reject H_0.
15. The test statistic is $Z_0 = \frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}}$. The critical region in terms of its observed value $z_0 = \frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}}$ is $|\frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}}| > z_{\alpha/2}$. Thus, the critical region in terms of $\bar{x}$ is

$$\overline{x} < \mu_0 - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \quad \text{or} \quad \overline{x} > \mu_0 + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}.$$

Therefore the critical region is

$$\overline{x} < 5 - 1.96(0.5)/\sqrt{(8)} = 4.65 \quad \text{or} \quad \overline{x} > 5 + 1.96(0.5)/\sqrt{(8)} = 5.35$$

16.

$$n \geq \left[\frac{z_{.005}\,\sigma}{E}\right]^2 = \left[\frac{(2.576)\,(30)}{15}\right]^2 = 26.5$$

Thus, we will collect $n = 27$ observations.
17. Since the population is normal, then $(\overline{X} - \mu)/(S/\sqrt{n})$ has a t distribution with $\nu = n - 1 = 9$ degrees of freedom. Hence $c = t_{0.01,9} = 2.821$.
18. We want

$$P(4 < X < 5) = F(5) - F(4) = \left(1 - \frac{4}{5^2}\right) - \left(1 - \frac{4}{4^2}\right) = 0.09.$$

19.

$$E[X] = \sum_{(x,y)} x\, p_{XY}(x,y) = 1;\;\; E[X^2] = \sum_{(x,y)} x^2\, f(x,y) = 1.5 \;\;\Rightarrow\;\; \sigma_X^2 = 1.5 - 1^2 = 0.5$$

$$E[Y] = \sum_{(x,y)} y\, p_{XY}(x,y) = 1;\;\; E[Y^2] = \sum_{(x,y)} y^2\, p_{XY}(x,y) = 1.5 \;\;\Rightarrow\;\; \sigma_Y^2 = 1.5 - 1^2 = 0.5$$

$$E[XY] = \sum_{(x,y)} xy\, p_{XY}(x,y) = 1.25; \quad \sigma_{XY} = 1.25 - (1)(1) = 0.25$$

Thus, $\rho_{XY} = \sigma_{XY}/\sqrt{\sigma_X^2\,\sigma_Y^2} = 0.25/\sqrt{0.5^2} = 0.5$

20. X="number of particles in 10kg" has a Poisson distribution with mean $\lambda = (0.02)(10) = 0.2$. We want

$$P(X = 0) = e^{-0.2}\frac{(0.2)^0}{0!} = 0.819$$

21. We want

$$\begin{aligned} P(\overline{X} > 1.5) &\approx 1 - \Phi\left(\frac{1.5 - 1}{\sqrt{4/100}}\right) \quad \text{(by the CLT since } n \text{ is large)} \\ &= 1 - \Phi(2.50) = 0.0062 \end{aligned}$$