Sample Midterm 1 - solutions

Short Answer Questions

[4] 1. The probability density function of the weight (in kg) of packages delivered by a post office is

$$f(x) = \begin{cases} 70/(69\,x^2), & 1 < x < 70 \\ 0, & \text{otherwise} \end{cases}$$

(a) What is the probability that the weight of a package will be more than 45 kg?

(b) Determine the expected weight (in kg) of a package.

(c) Determine the standard standard of the weight (in kg) of a package.

(d) If the shipping cost is \$5.25 per kg, what is the mean shipping cost of a package.

solution: Let X be the weight (in kg) of package.

(a) We want

$$P(X > 45) = \int_{45}^{\infty} f(x)\,dx = \int_{45}^{70} 70/(69\,x^2)\,dx = 0.0081.$$

(b) The expected weight (in kg) of a package is

$$E[X] = \int_{\infty}^{\infty} x\,f(x)\,dx = \int_{1}^{70} 70/(69\,x)\,dx = 4.3101.$$

(c) We compute

$$E[X^2] = \int_{\infty}^{\infty} x^2\,f(x)\,dx = \int_{1}^{70} 70/69\,dx = 70.$$

The standard standard of the weight (in kg) of a package is

$$\sigma_X = \sqrt{V[X]} = \sqrt{E[X^2] - \mu_X^2} = \sqrt{70 - (4.3101)^2} = 7.1710.$$

(d) The mean shipping cost of a package (in \$) is $E[5.25\,X] = 5.25\,(4.3101) =$ \$22.6280.

[4] 2. Suppose that in a plant that manufactures integrated circuit chips, 15% of chips are defective, and let us select chips randomly from the output of the plant.
(a) If 10 chips are selected, what is the probability that at most one chip is defective?
(b) If 8 chips are selected, what is the probability that exactly two chips work (i.e. are not defective)?
(c) What is the probability that the third defective chip will be the nineteenth chip selected?
(d) How many chips would you expect to select when encountering the first defective chip?
(e) How many defective chips would you expect to find in a selection of 10 chips?

Solution: (a) We want $P(X \leq 1)$, where $X \sim B(n = 10; p = 0.15)$. Thus,

$$P(X = 1) = \binom{10}{0} p^0 (1-p)^{10} + \binom{10}{1} p^1 (1-p)^9 = 0.5443.$$

(b) We want $P(X = 2)$, where $X \sim B(n = 8; p = 0.85)$ (careful, prob of a **non**-defective is 0.85). Thus,

$$P(X = 2) = \binom{8}{2} p^0 (1-p)^8 = 0.0002304.$$

(c) We want $P(X = 3)$, where X has a negative binomial distribution with $r = 19$ and $p = 0.15$. We want

$$P(X = 3) = \binom{18}{2} 0.15^3 0.85^{19-3} = 0.03834.$$

(d) This is the mean of a geometric distribution, $\mu = 1/p = 1/0.15 = 6.667$.
(e) This is the mean of a binomial distribution, $\mu = np = 1.5$.

Multiple Choice Questions

Please enter your answers to the multiple choice question in the table provided on the first page.

[1] 1. Suppose that the joint distribution of the random variables X and Y is given by the following table:

x	y	$P(X = x, Y = y)$
3	0	1/45
4	0	2/45
5	0	5/45
3	1	8/45
4	1	6/45
5	1	7/45
3	2	3/45
4	2	9/45
5	2	4/45

What is the probability that X equals 5 given that Y equals 1?

A) 0.4667 B) 0.3111 C) 0.1556 D) 0.3333 E) 0.4375

Solution: $P(X = 5|Y = 1) = P(X = 5 \cap Y = 1)/P(Y = 1) = \frac{7/45}{8/45+6/45+7/45} = 7/21 = 0.3333.$

[1] 2. Consider a manufacturing process of engine valves. All valves are subject to a first grind. If a valve's thickness meets specifications, then it is ready for installation. If the thickness is too large, then the valve is reground. If the thickness is too small, then the valve is scrapped. Assume that after a first grind, 75% of the valves meet the specifications, 15% are reground and 10% are scrapped. Furthermore, assume that among the valves that are reground, 95% meet the specifications, and 5% are scrapped. Find the probability that a valve will meet the specifications (after either the first or the second grind).

A) 0.7975 B) 0.8925 C) 0.7125 D) 0.9225 E) 0.8403

Solution: Let us consider the following partition of the sample space: A_1, A_2, A_3, were A_1 is the event that the valve meet the specification after the first grind, A_2 is the event that the valve is reground and A_3 is the event that the valve is scrapped after the first grind. Let M be the event that the valve meets the specifications. We have $P(A_1) = 0.75$, $P(A_2) = 0.15$, $P(A_3) = 0.10$, $P(M|A_1) = 1$, $P(M|A_2) = 0.95$ and $P(M|A_3) = 0$. By the total probability rule, the probability that a valve will meet the specifications is

$$\begin{aligned} P(M) &= P(M|A_1)P(A_1) + P(M|A_2)P(A_2) + P(M|A_3)P(A_3) \\ &= (1)\,(0.75) + (0.95)\,(0.15) + 0\,(0.1) \\ &= 0.8925 \end{aligned}$$

[1] 3. Assume that each of your calls to a popular radio station has a probability of 0.02 of connecting (that is not obtaining a busy signal). Assume that your calls are independent. What is the probability that your first call that connects is your tenth call?

A) 0.0741 B) 0.0167 C) 0.817 D) 0.183 E) 0.1677

Solution: Let X be the number of calls you need to get connected. Then X has a geometric distribution with $p = 0.98$. We want $P(X = 10) = (0.02)^9\,(0.98) = 0.0167$

[1] 4. We write a computer program to generate 8-character passwords. Each character is equally likely to be any of 26 letters or 10 digits (i.e. one of 36 possible characters). Assume that there is a condition that the password must have a least one digit. What is the probability that our password generator will produce a valid password?

A) 0.074 B) 0.722 C) 0.948 D) 0.052 E) 0.926

solution: By the multiplication principle, the password generator can produce 36^8 different passwords, each being equally likely of being generated. Among these passwords, there are 26^8 that do not contain any digits. So the probability that we will generate an invalid password is $26^8/36^8 = 0.074$. In other words, it will produce a valid password with a probability of $1 - 0.074 = 0.926$.

[1] 5. A programmer is having his work assessed by having 100 randomly selected lines of her code scrutinized. The random variable X, representing the number of errors in the 100 lines of code, has the following probability distribution:

x	2	3	4
$P(X = x)$	0.5	0.3	0.2

If the employer pays the programmer a bonus of 500 dollars minus 3.5 times of number of errors found, then what is the expected bonus for this programmer, in dollars?

A) 490.55 B) 400 C) 333.33 D) 425 E) 347.23

Solution: We want

$$E[500 - 3.5X] = 500 - 3.5E[X] = 500 - 3.5(2.7) = 490.55$$

where $E[X] = 2\,(0.5) + 3\,(0.3) + 4(0.2) = 2.7$. The answer is (A).

[1] 6. Each item are tested by two quality control engineers. If a flaw is present, then the first inspector will identify it with a probability of 0.95, while the second inspector will identify it with a probability of 0.75. Conditional on a flaw being present, assume that the conclusions of the inspectors are independent of each other. If both inspectors test an item with a flaw, what is the probability that at least one of the two quality control engineers will identify the flaw?

A) 0.7125 B) 0.9957 C) 0.9875 D) 0.9753 E) 0.8753

Solution: Let D_i be the event that the ith inspector identifies the flaw for $i = 1, 2$. We want

$$\begin{aligned} P(D_1 \cup D_1) &= P(D_1) + P(D_2) - P(D_1 \cap D_2) \\ &= 0.95 + 0.75 - (0.95)(0.75) \quad \text{by independence} \\ &= 0.9875 \end{aligned}$$