MAT 2377 A (Winter 2017)

Assignment 3
Solution
There are 3 questions

Remark: *Please only grade questions 1 and 2.*

1. (a) Firstly, we determine the marginal probability mass functions for X et Y

$$p_X(x) = \sum_y p_{XY}(x,y) = \begin{cases} 15/50, & x=-2 \\ 25/50, & x=0 \\ 8/50, & x=2 \end{cases} \quad \text{and} \quad p_Y(y) = \sum_x p_{XY}(x,y) = \begin{cases} 17/50, & y=-5 \\ 15/50, & y=0 \\ 18/50, & y=5 \end{cases}$$

[1] So that,

$$\mu_X = E[X] = \sum_{x \in R_X} x\, p_X(x) = (-2)\left(\frac{15}{50}\right) + 0\left(\frac{25}{50}\right) + 2\left(\frac{8}{50}\right) = \frac{-14}{50} = -0.28$$

[1] and

$$\mu_Y = E[Y] = \sum_{y \in R_Y} y\, p_Y(x) = (-5)\left(\frac{17}{50}\right) + 0\left(\frac{15}{50}\right) + 5\left(\frac{18}{50}\right) = \frac{5}{50} = 0.1.$$

[1] With the joint probability mass function, we compute

$$E[XY] = \sum_{(x,y)\in R_{XY}} x\,y\,p_{XY}(x,y) = (-2)\times(-5)\times\frac{7}{50}+0+(-2)\times 5\times\frac{3}{50}+0+0+0+2\times(-5)\times\frac{1}{50}+0+2\times 5\times\frac{3}{50} = 60/50 = 1.2.$$

[1] Therefore the covariance between X and Y is

$$\text{Cov}[X,Y] = E[XY] - \mu_X\mu_Y = 1.2 - (-0.28)(0.1) = 1.228$$

[1] (b) Since $\text{Cov}[X,Y] \neq 0$, we conclude that X and Y are **not independent** random variables.

(c) We have

$$E[X^2] = \sum_{x\in R_X} x^2 \times p_X(x) = (-2)^2\left(\frac{15}{50}\right) + 0^2\left(\frac{25}{50}\right) + 2^2\left(\frac{8}{50}\right) = \frac{92}{50}$$

$$E[Y^2] = \sum_{y\in R_Y} y^2 \times p_Y(y) = (-5)^2\left(\frac{17}{50}\right) + 0^2\left(\frac{15}{50}\right) + 5^2\left(\frac{18}{50}\right) = \frac{875}{50}$$

[1] it follows that,

$$V[X] = E[X^2] - \mu_X^2 = \frac{92}{50} - (-0.28)^2 = 1.7616$$

[1]

$$V[Y] = E[Y^2] - \mu_Y^2 = \frac{875}{50} - (0.1)^2 = 17.49.$$

[2] Hence,

$$V[X+2Y] = V[X]+V[2Y]+2\text{Cov}[X,2Y] = V[X]+2^2V[Y]+2\times 2\text{Cov}[X,Y] = 1.7616+4(17.49)+4\,(1.228) = 76.6336$$

[2] and

$$\begin{aligned} V[7X-Y] &= V[7X]+V[-Y]+2\text{Cov}[7X,-Y] = 7^2V[X]+(-1)^2V[Y]+(2)(7)(-1)\text{Cov}[X,Y] \\ &= (49)(1.7616)+(1)(17.49)+(-14)(1.228) = 86.6164 \end{aligned}$$

[1] 2. (a)

$$P(X=-1) = p_{XY}(-1,-1) + p_{XY}(-1,0) = \frac{1}{20} + \frac{4}{20} = \frac{5}{20} = 0.25$$

[1]

$$P(X=-1|Y=-1) = \frac{P(X=1, Y=-1)}{P(Y=-1)} = \frac{p_{XY}(-1,-1)}{p_{XY}(-1,-1)} = 1.$$

[2] (b)

$$p_X(x) = \begin{cases} 1/20 + 4/20 = 5/20, & x = -1 \\ 2/20 + 7/20 + 2/20 = 11/20, & x = 0 \\ 3/20, & x = 1 \\ 1/20, & x = 2 \end{cases} \quad \text{and} \quad p_Y(y) = \begin{cases} 1/20, & y = -2 \\ 1/20, & y = -1 \\ 4/20 + 2/20 = 6/20, & y = 0 \\ 7/20 + 3/20 = 10/20, & y = 1 \\ 2/20, & y = 3 \end{cases}$$

[1] (c)

$$\mu_X = \sum_{x \in R_X} x\, p_X(x) = (-1)\left(\frac{5}{20}\right) + (0)\left(\frac{11}{20}\right) + (1)\left(\frac{3}{20}\right) + (2)\left(\frac{1}{20}\right) = 0$$

[1]

$$E[XY] = \sum_{(x,y) \in R_{XY}} xy \times p_{XY}(x,y) = (-1)(-1)\left(\frac{1}{20}\right) + 0 + 0 + 0 + 0 + (1)(1)\frac{3}{20} + 2(-2)\left(\frac{1}{20}\right) = 0.$$

[1] Therefore,

$$\text{Cov}[X,Y] = E[XY] - \mu_X \mu_Y = 0 - 0 \times \mu_Y = 0 - 0 = 0$$

[2] (d) From (a), we have $P(X=-1) = \frac{1}{4} \neq 1 = P(X=-1|Y=-1)$. It follows that, X et Y are **not independent** random variables.

Another way to show that X et Y are **not independent** random variables is to use the following counter example: for $(x,y) = (2,-2)$

$$p_X(2)\, p_Y(-2) = \frac{1}{20} \times \frac{1}{20} = \frac{1}{400} \neq \frac{1}{20} = p_{XY}(2,-2).$$

Hence, it is not true that $p_X(x)p_Y(y)$ is equal to $p_{XY}(x,y)$ for all $(x,y) \in R_X \times R_Y$. Consequently, X et Y are **not independent** random variables.

3. (a) We are going to use $\sum_{x=0}^{\infty}(1/2)^x = 1/(1-1/2) = 2$ and $\sum_{y=0}^{\infty}(1/2)^y = 1/(1-1/2) = 2$. By solving

$$\begin{aligned} 1 &= \sum_{x=0}^{\infty}\sum_{y=0}^{\infty} c\,(1/2)^x\,(1/2)^y = c\sum_{x=0}^{\infty}(1/2)^x \sum_{y=0}^{\infty}(1/2)^y = c\sum_{x=0}^{\infty}(1/2)^x\,(2) \\ &= 2\,c\sum_{x=0}^{\infty}(1/2)^x = 2\,c\,(2) = 4\,c \end{aligned}$$

we found that, $c = 1/4$.

(b) The range of the discrete random variable X is $R_X = \{0, 1, 2, \ldots\}$. We want to find

$$\begin{aligned} p_X(x) = P(X=x) &= \sum_{y \in R_Y} p_{XY}(x,y) = \sum_{y=0}^{\infty}(1/4)\,(1/2)^x\,(1/2)^y = (1/4)\,(1/2)^x \sum_{y=0}^{\infty}(1/2)^y \\ &= (1/4)\,(1/2)^x\,2 = (1/2)\,(1/2)^x. \end{aligned}$$

The range of the discrete random variable Y is $R_Y = \{0, 1, 2, \ldots\}$. We want to find

$$\begin{aligned} p_Y(y) = P(Y=y) &= \sum_{x \in R_X} p_{XY}(x,y) = \sum_{x=0}^{\infty}(1/4)\,(1/2)^x\,(1/2)^y = (1/4)\,(1/2)^y \sum_{x=0}^{\infty}(1/2)^x \\ &= (1/4)\,(1/2)^y\,2 = (1/2)\,(1/2)^y. \end{aligned}$$

(c) For all $(x, y) \in R_X \times R_Y$, we have

$$p_{XY}(x) = (1/4)(1/2)^x(1/2)^y = p_X(x)\,p_Y(y).$$

Hence, X et Y are independent random variables.

(d) Since X et Y are independent random variables, we conclude that $\mathrm{Cov}(X, Y) = 0$.

[**/20**]