MAT 2377 (Winter 2017)
Prof: Rachid Bentoumi

Assignment 5
Solution

Remark: *Please only grade questions 2, 4 and 5.* ***Total=15 points***

1. Let X be a random variable which denotes the lifetime (months) of a transistor.

   (a) Since $X \sim Exp(\lambda)$, $E[X] = \dfrac{1}{\lambda} = 5$. It follows that $\lambda = 1/5 = 0.2$. Now, we need to find

   $P(X > 7) = e^{-0.2(7)} = e^{-1.4} = 0.2465$ (because $P(X > x) = e^{-\lambda x}$).

   (b) We have to find $P(3 \leq X \leq 8) = F(8) - F(3) = (1 - e^{-0.2(8)}) - (1 - e^{-0.2(3)}) = 0.3469$.

   (c) We want $P(X > 4 + 3|X > 4) = P(X > 3) = e^{-0.2(3)} = 0.5488$.

2. The waiting time until the first call follows exponential distribution with parameter $\lambda = 0.2$.

   (a) Since $X \sim Exp(\lambda)$, $E[X] = \dfrac{1}{\lambda} = \dfrac{1}{0.2} = 5$ and $V[X] = \dfrac{1}{\lambda^2} = \dfrac{1}{0.2^2} = 25$.

   (b) We evaluate $P(X \leq 1) = F(1) = 1 - e^{-0.2(1)} = 0.1812$.

   (c) Let Y be the waiting time until the third calls. We may write $Y = X_1 + X_2 + X_3$ where, X_1, X_2 and X_3 are independent random variables following exponential distribution with parameter $\lambda = 0.2$. Therefore Y follows Erlang distribution with parameters $r = 3$ and $\lambda = 0.2$. Now we need to determine

   $$P(Y < 2) = 1 - \sum_{k=0}^{3-1} e^{-0.2(2)} \frac{(0.2(2))^k}{k!} = e^{-0.4}\left(1 + \frac{0.4}{1} + \frac{0.4^2}{2}\right) = 1 - 0.9920 = 0.008$$

   where we used $P(Y < y) = 1 - \sum_{k=0}^{r-1} e^{-\lambda y} \dfrac{(\lambda y)^k}{k!}$.

3. We have $X \sim N(\mu = 25, \sigma^2 = 16)$.

   (a) $P(X > 18) = 1 - P(X \leq 18) = 1 - P\left(\frac{X-\mu}{\sigma} \leq \frac{18-\mu}{\sigma}\right) = 1 - P\left(Z \leq \frac{18-25}{4}\right) = 1 - \Phi(-1.75) = 1 - (1 - \Phi(1.75)) = \Phi(1.75) = 0.9599$.

   (b) $P(27 < X < 35) = P\left(\frac{27-25}{4} < \frac{X-25}{4} < \frac{35-25}{4}\right) = P(0.5 < Z < 2.5) = \Phi(2.5) - \Phi(0.5) = 0.9938 - 0.6915 = 0.3023$.

   (c) $P(17 < X < 23) = P\left(\frac{17-25}{4} < \frac{X-25}{4} < \frac{23-25}{4}\right) = P(-2 < Z < -1) = \Phi(2) - \Phi(0.5) = 0.9772 - 0.6915 = 0.2857$.

   (d) $P(c - X \leq 4) = P(X \geq c - 4) = 1 - P(X < c - 4) = 1 - P\left(\frac{X-25}{4} < \frac{c-29}{4}\right) = 1 - \Phi(\frac{c-29}{4})$. Now, since $P(c - X \leq 4) = 0.873$ it follows that $1 - \Phi(\frac{c-29}{4}) = 0.872$. Consequently, $\Phi(\frac{c-29}{4}) = 1 - 0.872 = 0.128$. Hence,

   $$\frac{c-29}{4} = \frac{-1.14 + (-1.13)}{2} = -1.135$$

   so that, $c = 4(-1.135) + 29 = 24.46$.

4. We have $X \sim N(\mu = 12.05, \sigma^2 = 0.03^2)$.

   (a) $P(X < 12) = P\left(\frac{X-\mu}{\sigma} \leq \frac{12-\mu}{\sigma}\right) = P\left(Z \leq \frac{12-12.05}{0.03}\right) = P\left(Z \leq -1.667\right) = \Phi(-1.667) = 1 - \Phi(1.667) = 1 - \left(\frac{0.9515+0.9525}{2}\right) = 1 - 0.952 = 0.048$. Therefore, the proportion of cans contain less than 12 oz is 4.8%.

   (b) Here, $X \sim N(\mu, \sigma^2 = 0.03^2)$ and $P\left(X > 12\right) = 0.99$. Firstly,
   $P\left(X > 12\right) = 1 - P\left(X \leq 12\right) = 1 - P\left(\frac{X-\mu}{0.03} < \frac{12-\mu}{0.03}\right) = 1 - P\left(Z < \frac{12-\mu}{0.03}\right) = 1 - \Phi\left(\frac{12-\mu}{0.03}\right) = 0.99$. So that, $\Phi\left(\frac{12-\mu}{0.03}\right) = 0.01$. Therefore,

   $$\frac{12-\mu}{0.03} = \frac{-2.33 + (-2.32)}{2} = -2.325$$

   so that, $\mu = 0.03(2.325) + 12 = 12.0697$.

   (c) Here, $X \sim N(\mu = 12.05, \sigma^2)$ and $P\left(X > 12\right) = 0.99$. Firstly,

   $P\left(X > 12\right) = 1 - P\left(X \leq 12\right) = 1 - P\left(\frac{X-12.05}{\sigma} < \frac{12-12.05}{\sigma}\right) = 1 - P\left(Z < \frac{-0.05}{\sigma}\right) = 1 - \Phi\left(\frac{-0.05}{\sigma}\right) = 0.99$. So that, $\Phi\left(\frac{-0.05}{\sigma}\right) = 0.01$. Therefore,

   $$\frac{-0.05}{\sigma} = \frac{-2.33 + (-2.32)}{2} = -2.325$$

   so that, $\sigma = \frac{0.05}{2.325} = 0.0215$.

5. **(a) Sample mean**: $\bar{x} = \sum_{i=1}^{n} x_i = \frac{5.4+4.6+\cdots+5.8}{17} = 6.1764$.

   Sample median: we first order our data in increasing order ($y_1 \leq y_2 \leq \cdots \leq y_n$) as follows:

   ```
   2.6  3.5  3.5  3.5  4.2  4.4  4.6  4.6  5.4  5.8  5.8  7.2  8.9  8.9 10.3 10.3 11.5
   ```

   and then, we determine the rank (position) of the 50th percentile: $(n+1)50/100 = (17+1)50/100 = 9$. Here, $m = 9$ and $p = 0$. Consequently, $\tilde{x} = y_9 = 5.4$.

   (b) To find the first quartile Q_1, we determine the rank of the 25th percentile $(n+1)25/100 = (17+1)25/100 = 4.5$ Here, $m = 4$ and $p = 0.5$. Consequently, $Q_1 = y_4 + 0.5(y_5 - y_4) = 3.5 + 0.5(4.2 - 3.5) = 3.85$.

   For the third quartile Q_3, we determine the rank of the 75th percentile $(n+1)75/100 = (17+1)75/100 = 13.5$ Here, $m = 13$ and $p = 0.5$. Consequently, $Q_3 = y_{13} + 0.5(y_{14} - y_{13}) = 8.9 + 0.5(8.9 - 8.9) = 8.9$.

   (b) The range of our data is $r = y_n - y_1 = y_{17} - y_1 = 11.5 - 2.6 = 8.9$.

   The interquartile range (distance) is $IQR = Q_3 - Q_1 = 8.9 - 3.85 = 5.05$.