MAT 2377A (Winter 2017)
Prof: Rachid Bentoumi

Assignment 6
Deadline : Please submit in the dropbox at 585 King Edward before 7:00pm on Thursday 6 April, 2017.
There are 4 questions.

Please solve the following problems with a calculator authorized by the Faculty of Science (TI30, TI34, Casio fx-260 or Casio fx-300) :

1. Let X, Y, and Z be independent normal random variables, where $X \sim N(\mu_X = 1, \sigma_X^2 = 3)$, $Y \sim N(\mu_Y = 0, \sigma_Y^2 = 2)$, and $Z \sim N(\mu_Z = 5, \sigma_Z^2 = 1)$.
   Let $W = 3X - 6Y + 2Z$.

[2] (a) Find $E[W]$ and $V[W]$.

[2] (b) What is the distribution of W?

[2] (c) What is $P(7 < W < 9)$?

solutions:

(a)

$$E[W] = 3E[X] - 6E[Y] + 2E[Z] = 3 \times 1 + 0 + 2 \times 5 = 13$$

Since X, Y and Z are independent,

$$V[W] = 3^2 \times V[X] + (-6)^2 \times V[Y] + 2^2 \times V[Z] = 3^2 \times 3 + (-6)^2 \times 2 + 2^2 \times 1 = 103$$

(b) Since Since X, Y and Z are both independent and normal, W must also be normal with mean $E[W] = 13$ and variance $V[W] = 103$. That is $W \sim N(13, 103)$.

(c)

$$P(7 < W < 9) = P\left(\frac{7 - \mu_W}{\sigma_W} < \frac{W - \mu_W}{\sigma_W} < \frac{9 - \mu_W}{\sigma_W}\right) = P\left(\frac{7 - 13}{\sqrt{103}} < \frac{W - \mu_W}{\sigma_W} < \frac{9 - 13}{\sqrt{103}}\right)$$
$$= P\left(-0.59 < \frac{W - \mu_W}{\sigma_W} < -0.39\right) = \Phi(-0.39) - \Phi(-0.59) = .3483 - .2776 = 0.071$$

2. Let $X_1, X_2, \ldots, X_n$ be a random sampling of size $n = 60$ from a population with the following distribution: For every i, $P(X_i = x) = \frac{1}{6}$ if $x = 0, 1, 2, 3, 4, 5$ and $P(X_i = x) = 0$ otherwise. Let $Y = X_1 + X_2 \ldots + X_n$.

   (a) What is the population mean μ and population variance σ^2?
   (b) What is $E[Y]$ and $V[Y]$?
   (c) Find the approximate distribution of Y.
   (d) Find the approximate value of $P(Y > 160)$

solutions:

(a)

$$\mu = E[X_i] = 0 \times \frac{1}{6} + 1 \times \frac{1}{6} + 2 \times \frac{1}{6} + 3 \times \frac{1}{6} + 4 \times \frac{1}{6} + 5 \times \frac{1}{6} = 2.5$$
$$E[X_i^2] = 0^2 \times \frac{1}{6} + 1^2 \times \frac{1}{6} + 2^2 \times \frac{1}{6} + 3^2 \times \frac{1}{6} + 4^2 \times \frac{1}{6} + 5^2 \times \frac{1}{6} = 9.17$$

so

$$\sigma^2 = V[X_i] = E[X_i^2] - (E[X_i])^2 = 9.17 - 2.5^2 = 2.92$$

(b)
$$E[Y] = n \times \mu = 60 \times 2.5 = 150$$
Since $X_1, \ldots X_n$ are independent, we have
$$V[Y] = n \times \sigma^2 = 60 \times 2.92 = 175.2$$

(c) By the central limit theorem, Y is approximately normally distributed with mean $E[Y] = 150$ and variance $V[Y] = 175.2$. That is, $Y \sim N(150, 175.2)$, approximately. (This approximation is considered to be good since $n = 60 > 30$)

(d)
$$P(Y > 160) = 1 - P(Y \leq 160) = 1 - P\left(\frac{Y - \mu_Y}{\sigma_Y} \leq \frac{40 - \mu_Y}{\sigma_Y}\right) \approx 1 - \Phi\left(\frac{160 - \mu_Y}{\sigma_Y}\right)$$
$$= 1 - \Phi\left(\frac{160 - 150}{\sqrt{175.2}}\right) = 1 - \Phi(0.76) = 1 - 0.7764 = 0.2236$$

3. Let $x_1, x_2, \ldots, x_n$ be a random sampling of size $n = 100$ from a population with unknown mean μ but **known** variance $\sigma^2 = 0.5$. The sample mean is $\overline{x} = 8.3$.

[2] (a) Find a (two-sided) 95% confidence interval for the population mean μ.

[2] (b) Find a 99.9% upper-confidence bound for the population mean μ.

[2] (c) Find a 90% lower-confidence bound for the population mean μ.

[2] (d) Suppose we can now choose the number n of samples to collect. How many samples in total should we collect so that the error $E = |\overline{x} - \mu|$ is such that we will be 90% confident that the error will satisfy $E \leq 0.007$?

solutions:

(a) $\alpha = 0.05$, so $z_{\alpha/2} = z_{0.025} = t_{0.025,\infty} = 1.96$. Therefore, a 95% confidence interval for μ is
$$\overline{x} \pm z_{\alpha/2}\frac{\sigma}{\sqrt{n}} = 8.3 \pm 1.96\frac{\sqrt{0.5}}{\sqrt{100}} = 8.3 \pm 0.139 = [8.3 - 0.139, 8.3 + 0.139]$$

(b) $\alpha = 0.001$, so $z_\alpha = 3.090$. Therefore a a 99.9% upper-confidence bound for μ is
$$\mu \leq \overline{x} + z_\alpha\frac{\sigma}{\sqrt{n}} = 8.3 + 3.090\frac{\sqrt{0.5}}{\sqrt{100}} = 8.52$$
(note: answers such as "$(-\infty, 8.52]$" are also acceptable.)

(c) $\alpha = 0.1$, so $z_\alpha = 1.282$. Therefore a 90% lower-confidence bound for μ is
$$\mu \geq \overline{x} - z_\alpha\frac{\sigma}{\sqrt{n}} = 8.3 - 1.282\frac{\sqrt{0.5}}{\sqrt{100}} = 8.21$$
(note: answers such as "$[8.52, \infty)$" are also acceptable.)

(d) $\alpha = 0.1$, so $z_{\alpha/2} = z_{0.05} = 1.645$.
Therefore, we must choose
$$n \geq \left(\frac{z_{\alpha/2}\sigma}{E}\right)^2 = \left(\frac{1.645\sqrt{0.5}}{0.007}\right)^2 = 27612.5$$
so using $n = 27613$ (or any integer larger than that) would work.

4. Let $x_1, x_2, \ldots, x_n$ be a random sampling of size $n = 10$ from a normal population with unknown mean μ and **unknown** variance σ^2. The sample mean is $\overline{x} = -6$ and the *sample* variance is $s^2 = 0.3$. Find a (two-sided) 95% confidence interval for the population mean μ.

solution:

$\alpha = 0.05$, so $t_{\alpha/2,n-1} = t_{0.025,9} = 2.262$. Therefore, a 95% confidence interval for μ is

$$\overline{x} \pm t_{\alpha/2,n-1}\frac{s}{\sqrt{n}} = -6 \pm 2.262\frac{\sqrt{0.3}}{\sqrt{10}} = 8.3 \pm 0.392 = [8.3 - 0.392, 8.3 + 0.392]$$

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