Continuous Probability Distributions

Recall: The cumulative distribution function of a continuous random variable X is :

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)\, dt,$$

where f is the probability density function of X such that

$$f(x) = \frac{dF(x)}{dx}.$$

Example 33: Consider a Poisson process with parameter λ. Let T be the waiting time until the first outcome of this Poisson process.

(a) Prove that the cumulative distribution function of T is

$$F_T(x) = \begin{cases} 0, & x < 0 \\ 1 - e^{-\lambda x}, & x \geq 0 \end{cases}$$

solution:

$$F_T(x) = P(T \leq x) = 1 - P(T > x)$$

$= 1 - P($“the poisson process has zero outcomes in the interval [0,x]”$)$

= 1- P(Y=0)

where Y is Poisson distributed with mean $\mu = \lambda x$.

So

$$F_T(x) = 1 - P(Y = 0) = 1 - e^{-\lambda x}\frac{(\lambda x)^0}{0!} = 1 - e^{-\lambda x}$$

(b) Prove that the probability density function of T is

$$f_T(x) = \lambda\, e^{-\lambda\, x}, \quad x > 0.$$

solution:

$$f_T(x) = \frac{\mathrm{d}}{\mathrm{d}x} F_T(x) = \frac{\mathrm{d}}{\mathrm{d}x}(1 - e^{-\lambda\, x}) = \lambda\, e^{-\lambda\, x}$$

The Exponential Distribution

Definition: Let X be a continuous random variable with probability density function

$$f_X(x) = \lambda\, e^{-\lambda\, x}, \quad x > 0,$$

where λ is a positive real number. We say that X has exponential distribution with parameter λ.

Remarks:

- From Example 33, we can conclude that the waiting time T until the next outcome of the Poisson process with parameter λ is exponentially distributed with parameter λ.
- The mean and variance of an exponential random variable X with parameter λ is

$$\mu_X = E[X] = \frac{1}{\lambda} \qquad \text{and} \qquad \sigma_X^2 = V(X) = \frac{1}{\lambda^2}$$

- The exponential distribution satisfies the **memoryless** property : Let X be an exponential random variable and let s and t be positive real numbers. Then,

$$P\,(X > s + t \mid X > s\,) = P(X > t).$$

Example 34 : Suppose that the defects in a roll of aluminum foil occur according to a poisson process with parameter $\lambda = \frac{1}{3}$ defects per meter.

(a) Determine the value t such that there is a 90% probability of having zero defects in t meters.

solution: Let T be the distance until the first defect. Then T has exponential distribution with parameter $\lambda = \frac{1}{3}$.

Therefore, t is the number such that

$$\begin{aligned} 0.9 &= P(\text{“no defects in } t \text{ meters”}) = P(T > t) \\ &= 1 - P(T \leq t) = 1 - F_T(t) = 1 - \left[1 - e^{-\lambda t}\right] \\ &= e^{-\lambda x} = e^{-\frac{1}{3}t} \end{aligned}$$

Therefore, $e^{-\frac{1}{3}t} = 0.9$ and so

$$-\frac{1}{3}t = \ln(0.9)$$

and so

$$t = -3\ln(0.9) = 0.316$$

(b) We check a roll of aluminum and find that there is no error in the first 3 meters. What is the probability that there will be no defects in the next 3 meters of the roll?

solution: By the memoryless property,

$$P\left(T > 3 + 3 \,\middle|\, T > 3\right) = P(T > 3) = 1 - P(T \leq 3) = 1 - \left[1 - e^{-\lambda \times 3}\right] = e^{-\frac{1}{3} \times 3} = e^{-1}$$

The Erlang distribution

Definition: Let T_r be the waiting time until the r'th outcome of a Poisson process with parameter λ. Then we say that X has *Erlang distribution* with parameters λ and r.

Its cumulative distribution function is (for $x > 0$):

$$\begin{aligned} F_{T_r}(x) &= P(T_r \leq x) = 1 - P(T_r > x) \\ &= 1 - P(\text{“at most } r-1 \text{ outcomes in} \\ &\qquad\qquad \text{the interval } [0, x]\text{”}) \\ &= 1 - \sum_{k=0}^{r-1} \frac{e^{-\lambda x} (\lambda x)^k}{k!} \end{aligned}$$

Its mean and variance are

$$\mu = E[X] = \frac{r}{\lambda} \qquad \text{and} \qquad \sigma^2 = V(X) = \frac{r}{\lambda^2}$$

(Note: An Erlang-(r, λ) random variable is a sum of r independent exponential random variables, each with parameter λ.)

Example 35 : Suppose we can model the arrivals of airplanes at an airport with a Poisson process of parameter $\lambda = 6$ airplanes per hour.

(a) What is the expected waiting time until 10 airplanes arrive?

solution: Let T_r be the waiting time until r airplanes arrive. Then T_r has Erlang-$(r, \lambda = 6)$ distribution. Therefore,

$$E[T_{10}] = \frac{r}{\lambda} = \frac{10}{6} = \frac{5}{3}$$

(b) What is the probability that the time until the arrival of 4 airplanes is more than 30 minutes?
solution:

Since λ is given in hours, to be consistent we must use "0.5 hours" in place of "30 minutes":

$$P(T_4 > 0.5\,\text{hours}) = 1 - P(T_4 \leq 0.5) = 1 - \left[1 - \sum_{k=0}^{4-1} \frac{e^{-\lambda\times 0.5}(\lambda \times 0.5)^k}{k!}\right]$$
$$= \sum_{k=0}^{4-1} \frac{e^{-6\times 0.5}(6 \times 0.5)^k}{k!}$$

The Normal Distribution:

Definition: Let μ be a real constant and σ a real positive contstant ($\sigma > 0$). A continuos random variable X with probability density

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/(2\,\sigma^2)}, \quad -\infty < x < \infty$$

is called a **normal** random variable with parameters μ and σ^2.

Notation: We say that X has distribution $N(\mu, \sigma^2)$.

Suppose that X has distribution $N(\mu, \sigma^2)$, then its mean and variance are

$$E[X] = \mu \qquad \text{and} \qquad V(X) = \sigma^2$$

Definition: Let Z be a normal random variable with mean $\mu = 0$ and variance $\sigma^2 = 1$. Then we say that Z has the **standard normal distribution**. Its density is

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}, \quad z \in \mathbb{R}.$$

Its cumulative distribution function is

$$\Phi(z) = P(Z \leq z) = \int_{-\infty}^{z} \phi(t)\, dt = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$$

Remarks:

1. Values for $\Phi(z)$ are given in tables. (there is one blackboard)
2. ϕ is symmetric about the origin, so
   (a) $\Phi(0) = P(Z \leq 0) = 0.5$
   (b) $P(Z \leq -z) = P(Z \geq z)$

Example 36:
Suppose that Z has distribution $N(0,1)$. Determine

(a) $P(0.53 < Z < 2.06)$

solution:

$$P(0.53 < X < 2.06) = \Phi(2.06) - \Phi(0.53) = 0.9803 - 0.7019 = 0.2784$$

(b) $P(-2.63 \leq Z \leq -0.51)$

solution:

$$P(-2.63 \leq Z \leq -0.51) = \Phi(-0.51) - \Phi(-2.63) = 0.3050 - 0.0043 = 0.3007$$

(c) $P(Z > 1.96)$

solution:

$$P(Z > 1.96) = 1 - P(Z \leq 1.96) = 1 - \Phi(1.96) = 1 - 0.975 = 0.025$$

(d) c such that $P(-c < Z < c) = 0.95$

solution: We want the value of c such that

$$0.95 = P(-c < Z < c)$$

$$= \Phi(c) - \Phi(-c)$$

$$= \Phi(c) - [1 - \Phi(c)]$$

since the standard normal density function is symmetric about the $x = 0$. Therefore,

$$0.95 = 2\Phi(c) - 1$$

so

$$\Phi(c) = \frac{1.95}{2} = 0.975$$

So we need to find the probability value "0.975" in the body of the table and see what value of c this corresponds to. If we do this, we get that

$$c = 1.96$$

(e) c such that $P(Z > c) = 0.03$

solution: We want the value of c such that

$$0.03 = P(Z > c)$$

$$= 1 - P(Z \leq c)$$

$$= 1 - \Phi(c)$$

So

$$\Phi(c) = 1 - 0.03 = 0.97$$

From the table we have that $\Phi(1.88) = 0.9699 \approx 0.97$. Therefore,

$$c \approx 1.88$$

Note: The following theorem says that we can always transform a normal random variable into a *standard* normal random variable, with a linear transformation.

Theorem: Let X have distribution $N(\mu_X, \sigma_X^2)$. Let

$$Z = \frac{X - E[X]}{\sqrt{V(X)}} = \frac{X - \mu_X}{\sigma_X}.$$

Then Z has distribution $N(0, 1)$.

Proof: Let X have distribution $N(\mu_X, \sigma_X^2)$ and let $Z = (X - \mu_X)/\sigma_X$.

$$\begin{aligned} F_Z(z) &= P(Z \leq z) = P\left(\frac{X - \mu_X}{\sigma_X} \leq z\right) = P\left(X \leq z\,\sigma_X + \mu_X\right) \\ &= \int_{-\infty}^{z\,\sigma_X + \mu_X} \frac{1}{\sqrt{2\,\pi}\,\sigma} e^{(x-\mu_X)^2/(2\sigma_X^2)}\,dx \end{aligned}$$

Let $u = (x - \mu_X)/(\sigma_X)$, so $du = (1/\sigma_X)\,dx$. Therefore, if $x = z\,\sigma_X + \mu_X$, then $u = (z\,\sigma_X + \mu_X - \mu_X)/(\sigma_X) = z$. Hence,

$$F_Z(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\,\pi}} e^{-u^2/2}\,dx = \Phi(z).$$

Therefore, Z has distribution $N(0, 1)$.

Consequences of this theorem :

$$P\left(X \leq x\right) = P\left(\frac{X - \mu_X}{\sigma_X} \leq \frac{x - \mu_X}{\sigma_X}\right) = \Phi\left(\frac{x - \mu_X}{\sigma_X}\right).$$

and so,

$$P\left(a < X < b\right) = F_X(b) - F_X(a) = \Phi\left(\frac{b - \mu_X}{\sigma_X}\right) - \Phi\left(\frac{a - \mu_X}{\sigma_X}\right).$$

Example 37: The width (in cm) of an aluminum component is normally distributed with mean $\mu = 2$ cm and standard deviation $\sigma = 0.007$ cm.

(a) The tolerated limits are 1.988 cm to 2.012 cm. If a piece has width outside the tolerated limits, it is defective. Find the proportion of defective pieces.

solution:

$$P(X \notin [1.988, 2.012]) = 1 - P(1.988 \leq X \leq 2.012])$$
$$= 1 - P(\frac{1.988 - \mu}{\sigma} \leq \frac{X - \mu}{\sigma} \leq \frac{2.012 - \mu}{\sigma}])$$
$$= 1 - P(-1.71 \leq \frac{X - \mu}{\sigma} \leq 1.71])$$
$$= 1 - [\Phi(-1.71) - \Phi(1.71)]$$
$$= 1 - [0.9564 - 0.0436 = 0.0872$$

(b) Determine the width (in cm) that will be exceeded by 90% of the aluminum components.

solution: We want the value of c such that

$$0.9 = P(X > c)$$
$$= P(\frac{X - \mu}{\sigma} > \frac{c - \mu}{\sigma})$$
$$= 1 - P(\frac{X - \mu}{\sigma} \leq \frac{c - \mu}{\sigma})$$
$$= 1 - \Phi(\frac{c - \mu}{\sigma})$$

so

$$\Phi(\frac{c - \mu}{\sigma}) = 1 - 0.9 = 0.1$$

So from the table, we have that

$$\frac{c - \mu}{\sigma} = -1.28$$

and so

$$c = -1.28\sigma + \mu = -1.28 \times 0.07 + 2 = 0.727$$

Example 38: The lifetime of a battery is normally distributed with a mean of 16 hours and a standard deviation of 2 hours.

(a) What is the probability that the given battery will have a lifetime of 19 hours?

(b) Find the value of x such that 10% of the batteries have a lifetime of more than x hours.

solutions: The solutions to Example 38 are similar to those of example 37.