

## Discrete Probability Distributions

**Definition:** A *Bernoulli trial* is a random experiment with two possible results, which we will call “success” and “failure”. Let  $p$  be the probability that we get a “success” outcome.

With the Bernoulli trial, we can define a *Bernoulli random variable*  $I$ :

$$I = \begin{cases} 1, & \text{if the Bernoulli trial is a “success”} \\ 0, & \text{if the Bernoulli trial is a “failure”} \end{cases}$$

Its expectation and variance are

$$E[I] = 0 \cdot (1 - p) + 1 \cdot p = p$$

and

$$V[I] = E[I^2] - (E[I])^2 = (0^2(1 - p) + 1^2p) - p^2 = p(1 - p).$$

**Definition:** A *binomial experiment* is composed of a fixed number  $n$  of independent Bernoulli trials, each with the same probability of “success”  $p$ .

Let  $X$  be the number of successes of a binomial experiment composed of  $n$  independent Bernoulli trials, each having the same success probability  $p$ .

We say that  $X$  is a *binomial random variable* with parameters  $n$  and  $p$ . We also say “ $X$  is binomially distributed with parameters  $n$  and  $p$ .”

**Mean and Variance of Binomial random variables:**

$$E(X) = np \quad \text{and} \quad V[X] = np(1 - p).$$

*Proof:* We can write  $X$  as a sum of  $n$  independent Bernoulli trials with the same success probability  $p$ :

$$X = \sum_{i=1}^n I_i, \quad \text{where} \quad I_i = \begin{cases} 1, & \text{if the } i\text{'th Bernoulli trial is a “success”} \\ 0, & \text{otherwise} \end{cases}$$

Therefore,

$$E(X) = E\left(\sum_{i=1}^n I_i\right) = \sum_{i=1}^n E(I_i) = \sum_{i=1}^n p = np$$

and

$$V(X) = V\left(\sum_{i=1}^n I_i\right) = \sum_{i=1}^n V(I_i) \text{ (by independence)} = \sum_{i=1}^n p(1-p) = np(1-p).$$

**Remark:** If we sample without replacement, the population we sample from changes as we remove samples from the population, and therefore the probability of success can change as well. However, if the population is very large (which is often the case), then this change can be so small that we can consider it negligible. In this case, all  $n$  trials can be considered to be (approximately) independent with (approximate) probability of success  $p$ .

The probability mass function of a binomial random variable  $X$  with parameters  $n$  and  $p$  is

$$p_X(x) = \binom{n}{x} p^x (1-p)^{n-x}, \text{ for } x = 0, 1, 2, \dots, n.$$

**Example 29:** When manufacturing circuit blocks, a coating of paint must be applied. Suppose that 70% of the blocks receive an acceptable coating.

(a) Determine the probability that, among 15 blocks selected a random,

(i) at most 13 will have an acceptable coating;

**Solution:** Let  $X$  be the number of blocks with an acceptable coating. Then  $X$  is a binomial random variable with parameters  $p = 0.7$  and  $n = 15$ .

$$P(X \leq 13) = 1 - P(X > 13) = 1 - [p_X(14) + p_X(15)] = 1 - \left[ \binom{15}{14} 0.7^{14} (1 - 0.7)^{15-14} + \binom{15}{15} 0.7^{15} (1 - 0.7)^{15-15} \right] = 0.965$$

(ii) at least 1 will have an acceptable coating.

**Solution:**  $P(X \geq 1) = 1 - P(X < 1) = 1 - p_X(0) = 1 - \binom{15}{0} 0.7^0 (1 - 0.7)^{15} = 1 - 0.3^{15} = 0.99999985651093$

(b) What is the expected number of blocks with an acceptable coating?

**Solution:**  $E[X] = np = 15 \times 0.7 = 10.5$

(c) What is the standard deviation of the number of blocks with an acceptable coating?

**Solution:**  $V[X] = np(1 - p) = 15 \times 0.7 \times (1 - 0.7) = 3.15$

so

$$\sigma_X = \sqrt{V[X]} = \sqrt{3.15}$$

## Geometric distribution

**Definition :** Let  $X$  be the number of independent Bernoulli trials required to observe a success. Suppose that each Bernoulli trial has the same probability  $p$  of success. Then we say that  $X$  has a *geometric distribution* with parameter  $p$ .

1. The probability mass function of the geometric distribution is

$$p_X(x) = P(X = x) = (1 - p)^{x-1} p, \quad \text{pour } x = 1, 2, 3, \dots$$

2. Its mean is

$$\mu = E(X) = \frac{1}{p}$$

3. Its variance is

$$\sigma^2 = V(X) = \frac{1 - p}{p^2}$$

4. Its cumulative distribution function is

$$\begin{aligned} F(x) &= P(X \leq x) \\ &= 1 - P(X > x) \\ &= 1 - P(\text{"the first } x \text{ trials are failures"}) \\ &= 1 - (1 - p)^x, \end{aligned}$$

for  $x = 1, 2, 3, \dots$

**Recall from your calculus courses :** The following series is called a geometric series :

$$a + ar + ar^2 + ar^3 + \dots$$

**note:** The series converges if and only if  $|r| < 1$ .

$$a + ar + ar^2 + ar^3 + \dots = \frac{a}{1 - r}, \quad \text{if } |r| < 1.$$

Therefore,

$$\begin{aligned}
 E(X) &= \sum_{x=1}^{\infty} x (1-p)^{x-1} p \\
 &= p \sum_{x=1}^{\infty} x q^{x-1} \quad (\text{defining } q = 1-p) \\
 &= p \frac{d}{dq} \left( \sum_{x=1}^{\infty} q^x \right) \\
 &= p \frac{d}{dq} \left( \frac{q}{1-q} \right) \quad (\text{since this is a geometric series with } a = q, r = q) \\
 &= p \left( \frac{(1-q) - (-1)q}{(1-q)^2} \right) \\
 &= p \frac{1}{p^2} \quad (\text{since } q = 1-p) \\
 &= 1/p
 \end{aligned}$$

Using a similar argument with a second derivative we can prove that  $E(X^2) = (2-p)/p^2$ . Therefore,

$$V[X] = E[X^2] - (E[X])^2 = \frac{2-p}{p^2} - \left(\frac{1}{p}\right)^2 = \frac{1-p}{p^2}.$$

## Negative Binomial Distribution

**Definition :** Let  $X$  be the number of independent Bernoulli trials required to observe our first  $r$  successes. Assume that each trial has the same probability  $p$  of success. Then we say that  $X$  has a *negative binomial* distribution with parameters  $p$  and  $r$ .

1. The p.m.f. of the negative binomial distribution is

$$f(x) = P(X = x) = \binom{x-1}{r-1} (1-p)^{x-r} p^r,$$

for  $x = r, r+1, r+2, \dots$

2. Its mean is

$$\mu = E(X) = \frac{r}{p}$$

3. Its variance is

$$\sigma^2 = V(X) = \frac{r(1-p)}{p^2}$$

**Example 30:** Electronic components are shipped to a supplier in boxes of 10. The supplier will return a box to the manufacturer only if it contains more than one defective component. Suppose that 2.5% of the components are defective.

(a) What is the probability that the 10th box received by the supplier will be the first that is returned to the manufacturer? (a) What is the probability that the 10th box received by the supplier will be the first that is returned to the manufacturer?

**Solution:** First, notice that each component is an independent Bernoulli trial with “success” probability  $q = 0.025$  (here a “success” means that the component is defective). Let  $Y_i$  be the number of defective components in the  $i$ 'th box shipped. Then each  $Y_i$  is a binomial random variable with parameters  $q = 0.025$  and  $n = 10$ .

Let  $p$  be the probability that any given box is returned to the supplier. Then

$$\begin{aligned} p &= P(Y_i > 1) = 1 - P(Y_i \leq 1) = 1 - [P(Y_i = 0) + P(Y_i = 1)] \\ &= 1 - \left[ \binom{10}{0} q^0 (1-q)^{10} + \binom{10}{1} q^1 (1-q)^9 \right] \\ &= 1 - \left[ \binom{10}{0} 0.025^0 (1-0.025)^{10} + \binom{10}{1} 0.025^1 (1-0.025)^9 \right] \approx 0.246 \end{aligned}$$

Since the components are mutually independent of each other, the number of components  $Y_1, Y_2, Y_3, \dots$  in the different boxes must also be mutually independent of each other. Therefore each box is itself an independent Bernoulli trial, but this time with “success” probability  $p = 0.246$  (here a “success” means that the box is returned). Let  $X$  be the number of boxes sent until the first box is returned. Then  $X$  must be a geometric random variable with

parameter  $p = 0.246$ . Therefore,

$$\begin{aligned} P(\text{"10'th box is first box returned"}) &= P(X = 10) = p_X(10) = (1 - p)^{10-1}p \\ &= (1 - 0.246)^9 \times 0.246 \approx 0.0194 \end{aligned}$$

(b) What is the expected number of boxes that the manufacturer will have received when it first returns a box to the supplier ?

**Solution:**  $E[X] = \frac{1}{p} = \frac{1}{0.246} \approx 4.07$

(c) What is the probability that more than 10 boxes will be received by the supplier before the first box is returned?

**Solution:**  $P(\text{"more than 10 boxes received before first box returned"}) = P(X > 10) = 1 - P(X \leq 10) = 1 - F_X(10) = 1 - [1 - (1 - p)^{10}] = (1 - p)^{10} = (1 - 0.246)^{10} \approx 0.594$

(d) What is the probability that the 10'th box received by the manufacturer will be the third that is returned to the supplier.

**Solution:** Let  $Z$  be the third box returned to the supplier (i.e., our third "success"). Then  $Z$  is a negative binomial random variable with parameters  $p = 0.246$  and  $r = 3$ . Therefore,

$$\begin{aligned} P(\text{"10'th box received is the third that is returned"}) &= P(Z = 10) = p_Z(10) \\ &= \binom{10-1}{r-1} (1-p)^{10-r} p^r = \binom{10-1}{3-1} (1-0.246)^{10-3} 0.246^3 \approx 0.074 \end{aligned}$$

## The Poisson Distribution

We can construct a probability distribution with the Taylor series for  $e^\mu$ , which is

$$e^\mu = \sum_{x=0}^{\infty} \frac{\mu^x}{x!}.$$

in other words,  $\sum_{x=0}^{\infty} e^{-\mu} \frac{\mu^x}{x!} = 1$ .

**Poisson distribution:** Let  $\mu$  be a positive real number. The random variable  $X$  with the following probability mass function is called a *Poisson random variable* with mean  $\mu$ :

$$p_X(x) = e^{-\mu} \frac{\mu^x}{x!}, \quad \text{for } x = 0, 1, 2, \dots$$

### Remarks:

- The mean and variance of a Poisson random variable are

$$E[X] = \mu \quad \text{and} \quad V[X] = \mu.$$

*Proof:*

$$\begin{aligned} E(X) &= \sum_{x=1}^{\infty} x e^{-\mu} \frac{\mu^x}{x!} \\ &= \mu e^{-\mu} \left[ \sum_{x=1}^{\infty} x \frac{\mu^{x-1}}{x!} \right] \\ &= \mu e^{-\mu} \frac{d}{d\mu} \left[ \sum_{x=1}^{\infty} \frac{\mu^x}{x!} \right] \\ &= \mu e^{-\mu} \frac{d}{d\mu} [e^\mu] \quad (\text{since this is the Taylor series for } e^\mu) \\ &= \mu e^{-\mu} e^\mu \\ &= \mu \end{aligned}$$

Using a similar argument we can prove that  $E[X^2] = \mu + \mu^2$ . Alors,  $V[X] = E[X^2] - (E[X])^2 = \mu$ .

- We will see that the Poisson distribution is a good approximation of the binomial distribution when  $n$  is large and  $p$  is small. This will then lead us to an application of the Poisson distribution called the *Poisson process*.

### Poisson approximation of the Binomial distribution

Let  $X$  be a Binomial random variable with  $n$  large and  $p$  small. Then,

$$P(X = x) = \binom{n}{x} p^x (1 - p)^{n-x} \approx e^{-\mu} \frac{\mu^x}{x!},$$

where  $\mu = np$ .

**Recall from your calculus courses:** We can define the number  $e$  with the following limit :

$$e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n.$$

More generally, we can show that

$$e^x = \lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n.$$

*Proof of the Poisson approximation:* Suppose that  $np = \mu$  is constant as  $n$  goes to  $\infty$ . Then,  $p \rightarrow 0$ . Let  $X$  be a binomial random variable with parameters  $n$  and  $p$ . We have

$$\begin{aligned} p_X(x) &= \binom{n}{x} p^x (1 - p)^{n-x} \\ &= \frac{n!}{x!(n-x)!} p^x (1 - p)^{n-x} \\ &= \frac{n!}{x!(n-x)!} (\mu/n)^x (1 - \mu/n)^{n-x} \\ &= \frac{\mu^x n(n-1)(n-2) \cdots (n-x+1)}{x! n^x} \frac{(1 - \mu/n)^n}{(1 - \mu/n)^x} \\ &\xrightarrow{n \rightarrow \infty} e^{-\mu} \frac{\mu^x}{x!} \end{aligned}$$

since

$$\frac{n(n-1)(n-2) \cdots (n-x+1)}{n^x} = (1)(1-1/n)(1-2/n) \cdots (1-(x-1)/n) \xrightarrow{n \rightarrow \infty} 1,$$

$$(1 - \mu/n)^n \xrightarrow{n \rightarrow \infty} e^{-\mu} \quad \text{and} \quad (1 - \mu/n)^x \xrightarrow{n \rightarrow \infty} 1.$$

**Rule of thumb :**

The Poisson approximation is good when  $n \geq 20$  and  $p \leq 0.05$ .

The Poisson approximation is very good when  $n \geq 100$  and  $np \leq 10$ .

**Example 32 :** In a given city, 3% of all drivers get at least one parking ticket per year. Compute the probability that among 80 drivers (randomly chosen in this city) at least 3 of these drivers will each get at least one parking ticket in any given year.

Please compute this probability exactly, and also with the Poisson approximation.

**solution:**

Let  $X$  be the number of “successes” (i.e.,  $X$  is the number of drivers with at least 1 parking ticket).

Then  $X$  is binomial with parameters  $n = 80$  and  $p = 0.03$ .

So the exact probability is

$$\begin{aligned} P(X \geq 3) &= 1 - P(X < 3) = 1 - [P(X = 0) + P(X = 1) + P(X = 2)] \\ &= 1 - \left[ \binom{80}{0} 0.97^{80} + \binom{80}{1} 0.03^1 0.97^{79} + \binom{80}{2} 0.03^2 0.97^{78} \right] = 0.568 \end{aligned}$$

Since  $n = 80$  is large and  $p = 0.03$  is small, we can also compute a poisson approximation, with  $\mu = n \times p = 8 \times 0.03 = 2.4$ :

$$\begin{aligned} P(X \geq 3) &= 1 - P(X < 3) = 1 - [P(X = 0) + P(X = 1) + P(X = 2)] \\ &\approx 1 - \left[ e^{-2.4} \frac{2.4^0}{0!} + e^{-2.4} \frac{2.4^1}{1!} + e^{-2.4} \frac{2.4^2}{2!} \right] = 1 - 0.48 = 0.5697 \end{aligned}$$

(notice that the Poisson approximation 0.5697 is very close to the true value 0.568, so this approximation is indeed very good).

## Poisson Process

**Definition:** Consider a collection of outcomes of random events that fall in the interval of nonnegative real numbers  $\{t \in \mathbb{R} : t \geq 0\} = [0, \infty)$  (we can interpret the different values in this interval as different times). We say that this random collection of outcomes is a *Poisson process* with parameter  $\lambda$  if

1. The numbers  $N_A$  and  $N_B$  of outcomes that fall in any two disjoint intervals A and B are independent random variables;
2. The probability of observing exactly one outcome in any given interval of “very small” length  $\Delta t$  is approximately  $p = \lambda \times (\Delta t)$ .
3. The probability of observing more than one occurrence in any given interval of “very small” length is approximately 0.

**Remark:** The Poisson process is a model that we use to study the outcomes of random variables in time (or space).

**Theorem:** Let  $X$  be the number of outcomes of a Poisson process with parameter  $\lambda$  in a given interval of size  $t$ . Then the random variable  $X$  has a Poisson *distribution* with parameter  $\mu = \lambda t$ .

**Informal proof:** Consider a Poisson process with parameter  $\lambda$  on an interval of length  $t$ . Partition this interval into  $n$  sub-intervals of equal length  $\Delta t = \frac{t}{n}$ . If  $n$  is very large, then  $\Delta t$  is very small. So, in any one of the these subintervals, the probability of getting exactly one outcome is approximately  $p = \lambda \times \Delta t$ , the probability of getting exactly 0 outcomes is approximately  $1 - p = 1 - \lambda(\Delta t)$ , and the probability of getting more than one outcome is approximately zero. So the number  $X$  of raindrops in any given interval is approximately Bernoulli distributed with parameter  $p = \lambda \times \Delta t$ . Therefore,

$$P(X = x) \approx \binom{n}{x} p^x (1 - p)^{n-x} \approx e^{-np} \frac{(np)^x}{x!} = e^{-\lambda t} \frac{(\lambda t)^x}{x!}.$$

**Note:** This approximation for  $X$  becomes exactly true if we take the limit as  $n$  goes to infinity.

**Example:** Consider the raindrops that fall in a 1-meter by 1-meter square on the sidewalk over a 5-minute period. Now consider the positions of these raindrops along the “x-axis” of this square. Then the collection of these positions is (approximately) a Poisson process with some parameter  $\lambda$  ( $\lambda$  is bigger if it rains harder). The *total number* of raindrops that fall in this square is (approximately) Poisson *distributed* with mean  $\mu = \lambda$ .

Please explain why the Poisson process is a good approximation for these raindrops.

**Example 33 :** Defects in a cassette tape occur at a rate of 1.5 defects per millimeter. Suppose that the positions of these defects follows a Poisson process. Let  $X$  be the number of defects on a given 5mm section of the tape.

(a) What is the probability distribution of  $X$ ?

**solution:**

Let  $\lambda = 1.5 \frac{\text{defects}}{\text{mm}}$  be the rate of the Poisson process and let  $t = 5\text{mm}$  be the width of the section of tape.

Since the random variable  $X$  is the number of outcomes of a Poisson *process* with rate  $\lambda$  that fall in an interval of size  $t$ , the Theorem on the previous page tells us that  $X$  has Poisson *distribution* with mean  $\mu = \lambda \times t = 1.5 \frac{\text{defects}}{\text{mm}} \times 5\text{mm} = 7.5$  defects.

Therefore the probability mass function of  $X$  is

$$P(X = x) = p_X(x) = e^{-\mu} \frac{\mu^x}{x!} = e^{-7.5} \frac{7.5^x}{x!} \quad \text{for } x = 0, 1, 2, \dots$$

(b) Compute  $P(X = 0)$  and  $E[X]$ .

**solution:**

$$P(X = 0) = p_X(0) = e^{-7.5} \frac{7.5^0}{0!} = e^{-7.5}$$
$$E[X] = \mu = 7.5$$