Name____

Student Number____________________________________

Lecture section (L1 Dr. Nielsen, L2 Dr. Wang)_______________

University of Calgary
Schulich School of Engineering
Fall 2010 Final Examination

ENGG 407
Numerical Methods

December 17, 2010, 12-3pm

3 Hours Duration

1. Examination is closed book.
2. No calculators
3. You do not need to simplify the numerical expressions unless stated
4. All angles are in radians, for example sin(x), x is assumed to be in radians
5. Final exam counts for 45% of overall course grade.
6. Exam has 40 multiple choice questions. Write the answers on the separate bubble sheet as well as in the spaces provided in this exam booklet.
7. Exam has 4 written questions. Write answers in the space provided below each question.
8. Total marks for the exam is 100. Marks value of each question is indicated.

Question	Area	Value	Mark
1	Multiple choice	40	
2	Ordinary differential equations	18	
3	Numerical derivatives and integration	12	
4	ODE and differentiations	20	
5	Gauss quadrature integrations	10	
total		100	

EXAMINATION RULES AND REGULATIONS

STUDENT IDENTIFICATION

Each candidate must sign the Seating List confirming presence at the examination. All candidates for final examinations are required to place their University of Calgary I.D. cards on their desks for the duration of the examination. (Students writing mid-term tests can also be asked to provide identity proof.) Students without an I.D. card who can produce an acceptable alternative I.D., e.g., one with a printed name and photograph, are allowed to write the examination.

A student without acceptable I.D. will be required to complete an Identification Form. The form indicates that there is no guarantee that the examination paper will be graded if any discrepancies in identification are discovered after verification with the student's file. A Student who refuses to produce identification or who refuses to complete and sign the Identification Form is not permitted to write the examination.

EXAMINATION RULES

(1) Students late in arriving will not normally be admitted after one-half hour of the examination time has passed.
(2) No candidate will be permitted to leave the examination room until one-half hour has elapsed after the opening of the examination, nor during the last 15 minutes of the examination. All candidates remaining during the last 15 minutes of the examination period must remain at their desks until their papers have been collected by an invigilator.
(3) All inquiries and requests must be addressed to supervisors only.
(4) Candidates are strictly cautioned against:

- (a) speaking to other candidates or communicating with them under any circumstances whatsoever;
- (b) bringing into the examination room any textbook, notebook or memoranda not authorized by the examiner;
- (c) making use of calculators and/or portable computing machines not authorized by the instructor;
- (d) leaving answer papers exposed to view;
- (e) attempting to read other student's examination papers.

The penalty for violation of these rules is suspension or expulsion or such other penalty as may be determined.

(5) Candidates are requested to write on both sides of the page, unless the examiner has asked that the left hand page be reserved for rough drafts or calculations.
(6) Discarded matter is to be struck out and not removed by mutilation of the examination answer book.
(7) Candidates are cautioned against writing in their answer book any matter extraneous to the actual answering of the question set.
(8) The candidate is to write his/her name on each answer book as directed and is to number each book.
(9) A candidate must report to a supervisor before leaving the examination room.
(10) Answer books must be handed to the supervisor-in-charge promptly when the signal is given. Failure to comply with this regulation will be cause for rejection of an answer paper.
(11) If during the course of an examination a student becomes ill or receives word of a domestic affliction, the student should report at once to the supervisor, hand in the unfinished paper and request that it be cancelled. If physical and/or emotional ill health is the cause, the student must report at once to a physician/counsellor so that subsequent application for a deferred examination is supported by a completed Physician/Counsellor Statement form. Students can consult professionals at University Health Services or University Counselling Services during normal working hours or consult their physician/counsellor in the community.

Should a student write an examination, hand in the paper for marking, and later report extenuating circumstances to support a request for cancellation of the paper and for another examination, such a request will be denied.

(12) Smoking during examinations is strictly prohibited.

RO 94-01

1. Multiple choice questions (40 total, valued at one mark each)

[b] **1.** Which statement **best describes** an ordinary differential equation (ODE)?
a. most commonly encountered type of differential equations
b. only one independent variable in the differential equation
c. can have more than one independent variable but only one dependant variable
d. restricted to being linear
e. can have any number of dependent and independent variables

[b] **2.** What is the advantage of using Crout's method instead of Gaussian elimination method for solving a system of equations $Ax = b$?
a. If A and b are constant then the back-substitution of Crouts method only needs to be done once.
b. Two back substitutions are more computationally efficient than a Gaussian elimination with a back substitution
c. None, Gaussian elimination is always the most efficient
d. If A is close to being singular then Crout's method is more accurate
e. Crout's method does not require pivoting

[c] **3.** Which is a false statement regarding $\frac{ds}{dt} = s + 5 + 3\frac{d^2 s}{dt^2}$?
a. It is a linear second order ODE
b. It has an algebraic solution such that numerical integration is not required
c. It can only be solved by the Modified Euler method.
d. It is a second order DEQ
e. All of the above statements are false

For the **next three questions** the following applies. A differential equation of the form $\frac{ds}{dt} = st^4$ with an initial value (condition) of $s(1) = 1$ is solved with a step size of $h = 0.1$

[a] **4.** If the explicit Euler method is used then
a. s(t) would be underestimated for all $t > 1$
b. s(t) would be overestimated for all $t > 1$
c. s(t) would be precisely calculated as the explicit form of Eulers is used
d. The propagated truncation error would be negligible as the DE is linear in terms of s
e. None of the above

[a] **5.** After the first iteration of the explicit Euler method then:
a. $s = 1.1$
b. $s = 0.9$
c. $s = 0.1$
d. $s = 1.0001$
e. None of the above

[d] 6. If solutions were generated based on the explicit Euler (EE), modified Euler (ME) and the fourth order Runge Kutta (RK) methods of this DEQ then the order of the most accurate to the least accurate solution $s(t)$ would be:

a. EE, ME, RK
b. ME, RK, EE
c. RK, EE, ME
d. RK, ME, EE

[d] 7. The equation $\frac{d^2s}{dt^2} = st^2 + s^3\frac{ds}{dt}$
a. is an ODE solvable only by the modified Euler method as it involves a nonlinear term
b. is solvable only by using the fourth order Runge Kutta method as it is nonlinear
c. is not solvable as it involves higher order derivative
d. none of the above

[c] 8. It is desired to find the solution of a projectile that is governed by the DEQ $\frac{d^3y}{dt^2} = f\left(t, y, \frac{dy}{dt}\right)$. What is a minimal set of conditions that can lead to a unique solution, given that the DEQ is solved for $t_0 \le t \le t_f$:
a. $\left\{y(t_0), y^{(1)}(t_0)\right\}$
b. $\left\{y(t_0), y^{(1)}(t_0), y^{(2)}(t_0), y^{(3)}(t_0)\right\}$
c. $\left\{y(t_0), y^{(1)}(t_0), y^{(1)}(t_f)\right\}$
d. $\left\{y(t_0), y^{(1)}(t_0), y(t_f), y^{(1)}(t_f)\right\}$
e. none of the above

[d] 9. Which of the following is the order of increasing accuracy of numerical integration methods for the same given interval size.
a. rectangular method Gauss Seidel method, Guass quadrature method
b. rectangular method, secant method, Guass quadrature method
c. rectangular method, Guass quadrature method, trapezoid method,
d. rectangular method, trapezoid method, Guass quadrature method

[e] 10. Which of the following is a valid statement regarding Simpsons 3/8 rule:
a. It is the integration of a piecewise cubic polynomial curve fit over two adjacent integration panels.
b. It is the integration of a piecewise quadratic polynomial curve fit over two adjacent integration panels
c. It uses a function evaluation at 3/8ths of the distance from each end of the integration panel.
d. It uses a normalized integration panel width of 3/8 while Simpsons 1/3 rule uses a normalized integration panel width of 1/3
e. It is the integration of a piecewise cubic polynomial curve fit over three adjacent integration panels

[d] 11. A Taylor series approximation to $f\left(x\right)$ is given as

$$\widehat{f_N}(x_1)=\sum_{n=0}^{N} f^{(n)}\left(x_o\right)\frac{\left(x_1-x_o\right)^n}{n!}$$

Which of the following statements is true:

a. x_1 is the point of expansion and the truncation error varies as $\left|x_1-x_o\right|^N$

b. x_o is the point of expansion and the truncation error varies as $\left|x_1-x_o\right|^N$

c. x_o is the point of expansion and the truncation error varies as $\left|x_1-x_o\right|^{N-1}$

d. x_o is the point of expansion and the truncation error varies as $\left|x_1-x_o\right|^{N+1}$

e. x_1 is the point of expansion and the truncation error varies as $\left|x_1-x_o\right|^{N+1}$

[a] 12. Which of the following methods does not perform well in dealing with stiff ODEs?
a. Euler's explicit method
b. Runge Kutta methods
c. Predictor-corrector methods
d. Adams Bashfourth multipoint method
e. none of the above methods can satisfactorily solve a stiff ODE

[c] 13. The first order Taylor series of $f\left(x,y\right)=x^2y$ with an expansion point of $\left(x=1,y=1\right)$ is
a. $-1+x+y$
b. $1+2x+y$
c. $2x+y-2$
d. 1
e. none of the above

[b] 14. The absolute value of the machine epsilon in a floating point representation with 15 bits in the mantissa and 10 bits in the exponent is approximately
a. 2^{15}
b. 2^{-15}
c. 2^{10}
d. 2^{-10}

[b] 15. A curve fitting is done with 3 cubic segments and a quadratic segment. How many conditions are needed to determine the coefficients of the piecewise (spline) polynomials?
a. 3
b. 15
c. 16
d. 12
e. none of the above

[b] 16. Which of the following is NOT a suitable method for solving boundary value problems for second order ODEs?
a. Shooting methods
b. Simpsons 3/8 rule
c. Transform into a system of initial value problems
d. Finite difference methods

[d] 17. Which of the following procedures will typically result in the fastest convergence in root finding given $f(x)=0$?
a. The bisection method
b. The overlapping method based on the golden ratio
c. The false position (regula falsi) method
d. The update recursion of $x_{i+1}=x_i-f(x_i)/f^{(1)}(x_i)$
e. Crout's method to determine roots of a general polynomial function

[a] 18. What is/are the eigenvalue(s) of the following matrix $\begin{bmatrix}1 & 4 & 0\\ 0 & 2 & 0\\ 0 & 0 & 3\end{bmatrix}$?

a. 1,2,3
b. 1,2,3,4
c. 0,2,3
d. $\infty, 0, 1$
e. none of the above

[b] 19. Given a system of linear equations $\mathbf{Ax}=\mathbf{b}$, according to the LU decomposition method, where $\mathbf{A}=\mathbf{LU}$, what is the procedure for finding $\mathbf{x}$?
a. one back-substitution operation of $\mathbf{A}$ to find $\mathbf{x}$ directly
b. forward-substitution of **L** followed by back substitution of **U**
c. back-substitution of **U** followed by back substitution of **L**
d. none of the above

[b] 20. Given a bracket of [a, b] where $f(a)f(b)<0$, at least how many roots would be found within the bracket?
a. 0
b. 1
c. indeterminable
d. None of the above
e. **a** and **b** only

[c] 21. Given two bytes for machine representation of float point numbers. Assume each byte (8 bits) is allocated for the mantissa and exponent, respectively, where the most significant bits are reserved for representing their signs. What is the range of number representation using this scheme?

a. $\pm\, 2^{8} \bullet 10^{\pm 2^{8}}$
b. $\pm 2^{7} \bullet 10^{\pm 2^{7}}$
c. $\pm\, 2^{7} \bullet 2^{\pm 2^{7}}$
d. $2^{\pm 7} \bullet 2^{\pm 2^{15}}$

[e] 22. When using the overlap method to find the minimum of $f(x)$ within an initial bracket of $[1,4]$ and an initial section length of 2, what is the length of the bracket of uncertainty around the minimum after 10 iterations?
a. 3^{-10}
b. $3 \bullet 2^{-10}$
c. $3/10$
d. can't tell, depends on $f(x)$
e. $3 \bullet 0.667^{10}$

*** { $b = L(1-k)^{n} = 3(1-1/3)^{10} = 3 \bullet 0.667^{10}$ }

[a] 23. What is the expected function that satisfies $\dfrac{d^2y}{dx^2} = \dfrac{dy}{dx} = y(x)$?
a. e^{x}
b. x^{e}
c. $e^{x} + C,$ C is a constant
d. $y = x$
e. none of the above

[b] 24. The Gauss Seidel method for solving a linear system of equations $\mathbf{Ax} = \mathbf{b}$
a. Requires a pre-decomposition stage followed by a back substitution
b. Converges quickly when the magnitude of the diagonal components of $\mathbf{A}$ are large in comparison with all of the off diagonal terms
c. Is an efficient way of performing the Gaussian elimination of $\mathbf{A}$.
d. Does not converge when the dimension of $\mathbf{A}$ is too large
e. None of the above

[d] 25. Given that we have data points of (x_1, y_1), (x_2, y_2) and (x_3, y_3) what is a Lagrange polynomial suitable for curve fitting?
a. $ax^2 + bx + 1$
b. $\dfrac{(x-x_1)(x-x_2)(x-x_3)}{x_1 x_2 x_3}$
c. $\dfrac{(x-x_1)(x-x_2)}{(x_1-x_3)(x_1-x_3)}$

d. $\frac{(x-x_1)(x-x_3)}{(x_2-x_3)(x_2-x_1)}$

[d] 26. The number of potential solution(s) for a first order ODE without the initial value specified is:
a. 0
b. 1
c. uncertain
d. infinite
e. 3

[a] 27. The most fundamental and widely used theory for numerical methods is:
a. Taylor series
b. Newton's method
c. Gaussian method
d. Euler's method
e. Crout's method

[c] 28. A disadvantage of Newton's iterative method for root finding is that it:
a. Only applicable to force related problems
b. Requires the second order numerical derivative
c. requires the algebraic form of the derivative
d. only applicable when the Jacobian is used
e. none of the above

[a] 29. Which of the following is NOT a suitable method for numerical integration?

a. Euler's explicit method
b. Trapazoidal method
c. Simpson's polynomials
d. Gauss quadrature

[a] 30. Crout's method of LU decomposition is applied to the matrix

$A=\begin{bmatrix}1&0&0\\0&2&0\\0&4&3\end{bmatrix}$. What are the L and U matrices?

a. $L=\begin{bmatrix}1&0&0\\0&2&0\\0&4&3\end{bmatrix}$ and $U=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$

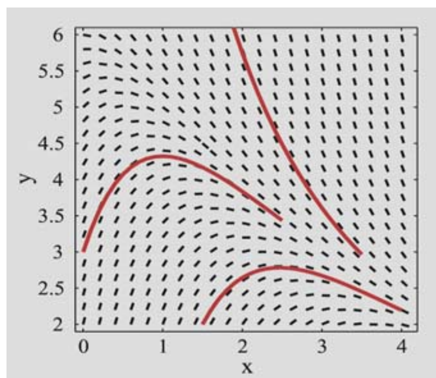
b. $L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 4 & 3 \end{bmatrix}$

c. $L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ and $U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 4/3 & 1 \end{bmatrix}$

d. $L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

e. none of the above

[d] 31. According to the figure given below for an ODE initial value problem, which of the following equation is NOT an valid initial value?



a. y(0) = 3
b. y(2) = 6
c. y(4) ≅ 2.2
d. all are valid initial values
e. none are valid initial values

[d] 32. The Euler's predictor-corrector method can be well described as follows *except*:
a. The predicator is determined by Euler's explicit method
b. The correctors are determined by the modified Euler's method
c. Each corrector is an iterative refinement using the modified Euler's method
d. It is computationally the fastest algorithm for ODE solving

[d] 33. In order to uniquely solve a system of n second order ODEs, how many initial values are needed?
a. n+1

b. n-1
c. n
d. 2n
e. problem specific

[d] 34. Identify if the given numerical difference formula $\frac{dy}{dx} = \frac{f(x_{i+1}) - f(x_{i-1})}{x_{i+1} - x_{i-1}}$ is:

a. A backward difference
b. A forward difference
c. A three-point backward difference
d. A central difference
e. A discrete form of a differential equation

[a] 35. In numerical optimization, the equivalence of the maximum and minimum at x*, f_{max}(x*) and f_{min}(x*), can be expressed as:
a. For all x*, f_{max}(x*) = - f_{min}(x*)
b. For all x*, f_{max}(x*) = f_{min}(-x*)
c. For all x*, f_{max}(-x*) = f_{min}(x*)
d. For some x*, f_{max}(x*) = - f_{min}(x*)

[d] 36 How can we tell that x_o maximizes $f(x)$?

a. $f(x_o) = 0$ and $f^{(1)}(x_o) = 0$

b. $f^{(1)}(x_o) = 0$ and $f^{(2)}(x_o) = 0$

c. $f^{(2)}(x_o) > 0$ and $f^{(1)}(x_o) = 0$

d. $f^{(2)}(x_o) < 0$ and $f^{(1)}(x_o) = 0$

e. $f^{(2)}(x_o) = 0$ and $f^{(1)}(x_o) < 0$

[c] 37. Given that a deterministic curve fit is required through 20 points what is the best and most practical approach?
a. Use Lagrange interpolation function
b. Use an interpolation polynomial of $f(x) = \sum_{n=0}^{20} a_n x^n$
c. Use multiple lower order splines
d. Use a least squares regression curve fit
e. Use the secant method

[c] 38. A least squares regression curve fit is applied to a set of N points of (t_1, x_1) (t_2, x_2) $\cdots$ (t_N, x_N). The regression function is $f(t) = C$, C is a constant. Which of the following is true:

a. C can only be determined if N=1

b. $C = \frac{1}{N}\sum_{i=1}^{N} x_i^{\,2}$

c. $C = \frac{1}{N}\sum_{i=1}^{N} x_i$

d. Insufficient information to determine the regression curve fit

e. $C = 1$

[c] **39.** In determining a numerical derivative with the interval between data samples h, what is a true statement regarding h.

a. To obtain the maximum numerical accuracy, h should be set at the machine epsilon

b. If h is decreased, the truncation error will decrease but the roundoff error will increase

c. If h is increased, the truncation error will increase but the roundoff error will decrease

d. h should be small to avoid numerical instability

e. h only affects truncation error and not roundoff error

[b] **40.** For a polynomial regression curve fit, the following is true:

a. Low order polynomials should be avoided as they are inaccurate

b. selecting too high a polynomial order will result in erroneous fitting of the noise or distortion in the data

c. only cubic polynomials should be used

d. coefficients can be efficiently determined using low order Lagrange polynomials

e. none of the above

2. (18) Differential Equations

Given $\frac{dy}{dx} = y - x,$ with initial value $y(0) = 1,$ and $h = 1.0,$ $0 \le x \le 2,$ solve the following initial value problems:

a) (5) Use *Euler's explicit* method to calculate the numerical solutions of *y(x)* for *x* = [0, 1, 2] and derive the equation of *y(x)*.

(ans)

$$y_{i+1} = y_i + f(x_i, y_i)h = y_i + h(y_i - x_i) = 2y_i - x_i$$

(1) $x_1 = 0,\ y_1 = 1$

(2) $x_2 = 1,\ y_2 = 2y_1 - x_1 = 2$

(3) $x_3 = 2,\ y_3 = 2y_2 - x_2 = 2 \bullet 2 - 1 = 3$

and

$$y(x) = x + 1$$

b) (5) Find the numerical solutions of *y(x)* for *x* = [0, 1, 2] by the *modified Euler's method*, where the result obtained in question (a) may be reused.

(ans)

$$\begin{cases} y_{i+1}^{Eu} = y_i + f(x_i, y_i)h = 2y_i - x_i \\ y_{i+1} = y_i + \dfrac{f(x_i, y_i) + f(x_{i+1}, y_{i+1}^{Eu})}{2} h = y_i + 0.5[(y_i - x_i) + (y_{i+1}^{Eu} - x_{i+1})] \end{cases}$$

(1) $x_1 = 0,\ y_1 = 1$

(2) $x_2 = 1,$

$$y_2 = y_1 + 0.5[(y_1 - x_1) + (y_2^{Eu} - x_2)] = 1 + 0.5[(1 - 0) + (2 - 1)] = 2$$

(3) $x_3 = 2,$

$$y_3 = y_2 + 0.5[(y_2 - x_2) + (y_3^{Eu} - x_3)] = 2 + 0.5[(2 - 1) + (3 - 2)] = 3$$

c) (8) Assume $\frac{d^2 g}{dt^2} = g,$ with boundary values $g(0) = 1,$ and $g(2) = 7.3891,$

(i) Express the ODE as a system of IVPs (first order ODE's);

(ii) Derive the equation of *g(t)* given $\frac{dg}{dt} = \frac{dw}{dt}$;

(iii) Calculate *g(t)* with *t* = [0, 1, 2] where *h* = 1.

(ans)

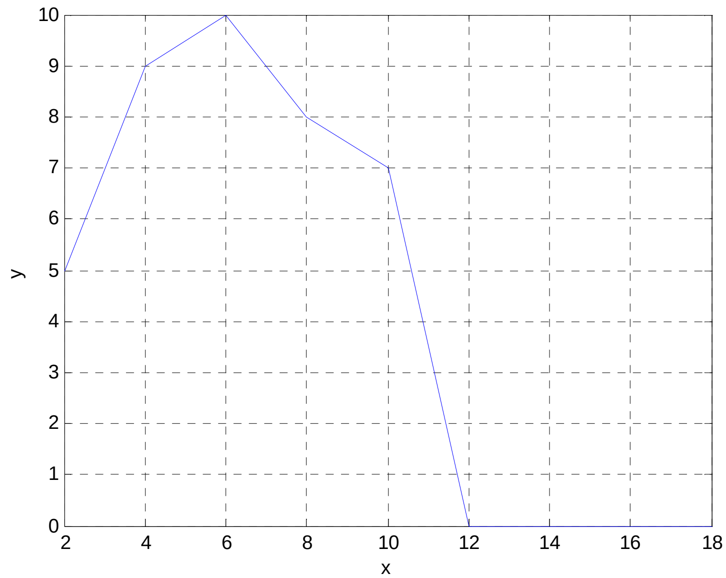
(i) $\frac{d^2g}{dt^2} = g,\ g(0) = 1,\ \text{and}\ g(2) = 7.3891 \Rightarrow \begin{cases} \frac{dg}{dt} = w,\ g(0) = 1 \\ \frac{dw}{dt} = g,\ w(0) = 1 \end{cases}$

(ii) $\frac{d^2g}{dt^2} = g,\ g(0) = 1,\ g(2) = 7.3891,\ \text{and}\ \frac{dg}{dt} = \frac{dw}{dt} \Rightarrow g(t) = e^t$

(iii) For $t = [0,1,2],\ g(t) = [1, e, e^2]$

3. (12) Numerical Derivatives and Integration

A set of data has been collected from an experiment as follows: *x* = [2, 4, 6, 8, 10, 12, 14, …, 10000] and *y* = [5, 9, 10, 8, 7, 0, 0, … , 0] as shown below.



a) (5) Determine where the curve's derivatives reached the maximum, minimum, and the turning point using the *central finite numerical difference* method.

[Answer (a)]

$$\frac{dy}{dx} = \frac{f(x_{i+1}) - f(x_{i-1})}{x_{i+1} - x_{i-1}}$$

$$= 0.5[f(4) - f(2), f(5) - f(3), f(6) - f(4), f(7) - f(5), f(8) - f(6), f(9) - f(7),$$
$$f(10) - f(8), f(11) - f(9), f(12) - f(10), \ ...]$$
$$= 0.5[4, 2.5, 1, -0.5, -2, -1.5, -1, -4.5, -7, \ 0, ..., 0]$$
$$= [2, 1.25, 0.5, -0.25, -1, -0.75, -0.5, -2.25, -3.5, \ 0, ..., 0]$$

That is:

$$D_{\max} = \frac{dy(3)}{dx} = 2.0$$

$$D_{\min} = \frac{dy(11)}{dx} = -3.5$$

$$D_o = \frac{dy(6)}{dx} \approx 0$$

b) (3) Calculate the numerical integration of this curve using the *midpoint method* for I_1 with h = 2.

[Answer (b)]

$$I_1 = h\sum_{i=1}^{n} f(\frac{x_i + x_{i+1}}{2})$$
$$= 2\sum_{i=1}^{6} f(\frac{x_i + x_{i+1}}{2})$$
$$= 2(f(3) + f(5) + f(7) + f(9) + f(11))$$
$$= 2(7 + 9.5 + 9 + 7.5 + 3)$$
$$= 72$$

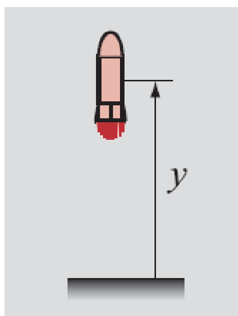
c) (4) Calculate the numerical integration of this curve again using the *trapezoid method* for I_2 with h = 1.

[Answer (c)]

$$I_2 = \frac{h}{2}[\hat{f}(a) + \hat{f}(b)] + h\sum_{i=2}^{n\text{-}1} \hat{f}(x_i)$$
$$= 0.5(5 + 0) + \sum_{i=3}^{11} f(x_i)$$
$$= 2.5 + (7 + 9 + 9.5 + 10 + 9 + 8 + 7.5 + 7 + 3)$$
$$= 72.5$$

4. (20) ODE and Differentiations

An experimental rocket having an initial weight of 3000 lb (including 2400 lb of fuel), and initially at rest, is launched vertically upward. The rocket burns fuel at a constant rate of 80 lb/s, which provides a constant thrust, T, of 8000 lb. The instantaneous weight of the rocket is $w(t) = 3000 - 80t$ lb. The drag force, D, experienced by the rocket is given by $D = 0.005g(\frac{dy}{dt})^2$ lb, where y is distance in ft, and g = 32.2 ft/s^2. According to Newton's law, the equation of motion for the rocket is given by:



$$\frac{w}{g}\frac{d^2y}{dt^2} = T - w - D, y(0) = 0, \frac{dy(0)}{dt} = 0$$

a) (5) Reduce the second-order ODE problem to a system of two first-order ODEs with proper initial values.

[Answer (a)]

$$\frac{dy}{dt} = v,\ y(0) = 0$$

$$\frac{dv}{dt} = \frac{g}{w}(T - w - 0.008gv^2),\ v(0) = 0$$

b) (6) Describe the function of velocity of the rocket based on the results of Part (a). Assume the observation of the positions of the rocket in the first 3 seconds at t = [0, 1, 2, 3] (s) are y = [0, 30, 100, 180] (ft), numerically determine the instantaneous velocity $v(t)$ of the rocket using the two-point backward Taylor series differentiation method.

[Answer (b)]

$$v(t) = \frac{dy}{dt}$$

According to 2B Taylor series diferentiation: $v(i) = \frac{y(t_i) - y(t_{i-1})}{h}, i = 1, 2, 3$

Obtain: $v(0) = 0$

$v(1) = 30 - 0 = 30$

$v(2) = 100 - 30 = 70$

$v(3) = 180 - 100 = 80$

c) (6) Express the function of the acceleration of the rocket based on the results of Part (a). Given the same observation of the positions of the rocket during t = [0, 1, 2, 3] (s), numerically determine the instantaneous acceleration *a(t)* of the rocket using the two-point backward Taylor series differentiation method. The results obtained in Part (b) may be reused.

[Answer (c)]

$$a(t) = \frac{dv}{dt} = \frac{g}{w}(T - w - 0.008gv^2), v(0) = 0$$

According to 2B Taylor series diferentiation: $\mathrm{a}(i) = \frac{v(t_i) - v(t_{i-1})}{h}, i = 1, 2, 3$

Obtain: $a(0) = 0$

$a(1) = 30 - 0 = 30$

$a(2) = 70 - 30 = 40$

$a(3) = 80 - 70 = 10$

d) (3) In order to improve the accuracy of the numerical differentiations in Parts (b) and (c), what approaches and/or means you may suggest?

[Answer (d)]

a) To reduce the *h*
b) To use a higher-order method

5. (10) Gauss Quadrature Integration

a. (5) Numerically integrate the function $f(x) = \sin(x^2)$ over the range of $0 \le x \le 2$ using the two-point Gauss quadrature method. Assume that two equal width panels are used over this interval. You do not have to evaluate the final expression.

(ans a)

$$I = \int_a^b f(x)dx \simeq \int_{-1}^{1} f\left(\frac{t \bullet (b-a)+a+b}{2}\right)\left(\frac{(b-a)}{2}\right)dt \simeq \sum_{i=1}^{n} C_i \hat{f}\left(\frac{t_i \bullet (b-a)+a+b}{2}\right)$$

$$I_g = \int_0^2 \sin(x^2)dx \simeq \int_{-1}^{1} \sin\left(\frac{2t+2}{2}\right)^2 dt = \int_{-1}^{1} \sin(t+1)^2 dt$$

$$= \sin\left(1-\frac{1}{\sqrt{3}}\right)^2 + \sin\left(1+\frac{1}{\sqrt{3}}\right)^2$$

b. (5) Repeat part **a** for $f(x) = x + x^2 + x^3$ given $-1 \le x \le 1$. You do not have to numerically evaluate the final expression.

(ans b)
Recall that the gauss quadrature is designed to be exact for polynomial integrands up to a cubic. Hence the answer is simply

$$\int_{-1}^{1} \left(x + x^2 + x^3\right)dx = \int_{-1}^{1} xdx + \int_{-1}^{1} x^2dx + \int_{-1}^{1} x^3dx$$

$$= 0 + \frac{2}{3} + 0$$

$$= \frac{2}{3}$$

Aid Sheet

Derivative Notation $f^{(n)}(x) = \dfrac{d^n f(x)}{dx^n}$

Taylor expansion Formula $\widehat{f_N}(x) = \sum_{n=0}^{N} \dfrac{1}{n!} f^{(n)}(x_o)(x - x_o)^n$

Simpson's $\frac{1}{3}$ method $I = \dfrac{h}{3}\left(f(x_0) + 4f(x_1) + f(x_2) \right)$

Simpson's $\frac{3}{8}$ method $I = \dfrac{3h}{8}\left(f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3) \right)$

2-point Gauss Quadrature integration

$$I = \int_a^b f(x)dx \simeq \int_{-1}^{1} f\left(\frac{t \bullet (b-a) + a + b}{2}\right)\left(\frac{(b-a)}{2}\right)dt \simeq \sum_{i=1}^{2} \hat{f}\left(\frac{t_i \bullet (b-a) + a + b}{2}\right) = \hat{f}\left(-\frac{1}{\sqrt{3}}\right) + \hat{f}\left(\frac{1}{\sqrt{3}}\right)$$

The midpoint integration method $I_1 = h\sum_{i=1}^{n} f\left(\dfrac{x_i + x_{i+1}}{2}\right)$

The trapezoid integration method $I_2 = \dfrac{h}{2}[f(a) + f(b)] + h\sum_{i=2}^{\mathbf{n-1}} f(x_i)$

2-point backward Taylor series differentiation $y(i) = \dfrac{y(t_i) - y(t_{i-1})}{h}$

The central finite difference method $\dfrac{dy}{dx} = \dfrac{f(x_{i+1}) - f(x_{i-1})}{x_{i+1} - x_{i-1}}$

Euler's explicit method $y_{i+1} = y_i + f(x_i, y_i)h$

The modified Euler's method $y_{i+1} = y_i + \dfrac{f(x_i, y_i) + f(x_{i+1}, y_{i+1}^{Eu})}{2}h$

Least squares regression $\mathbf{M} = \mathbf{AP}$ $\quad \mathbf{P} = \left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T\mathbf{M}$

Taylor Series – Vector Form

$$\hat{f}_1(\mathbf{x}) = f(\mathbf{x}_0) + \mathbf{J}(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0)^{\mathrm{T}}$$

$$\hat{f}_2(\mathbf{x}) = f(\mathbf{x}_0) + \mathbf{J}(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0)^{\mathrm{T}} + \frac{1}{2}(\mathbf{x} - \mathbf{x}_0)^{\mathrm{T}}\mathbf{H}(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0)$$

where

$$\mathbf{J} = \left[\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, ..., \frac{\partial f}{\partial x_N}\right] \quad \mathbf{H} = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_N} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_N} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial^2 f}{\partial x_N \partial x_1} & \frac{\partial^2 f}{\partial x_N \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_N^2} \end{bmatrix}$$